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On the Cauchy problem for a reaction–diffusion equation with a singular nonlinearity

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Abstract

We consider the following Cauchy problem with a singular nonlinearity

$$(P)\qquad \begin{cases} u_t = \Delta u - u^{-\nu}, & x \in \mathbf{R}^n,\ t > 0,\ \nu > 0, \\ u|_{t=0} = \phi \in C_{LB}(\mathbf{R}^n) \end{cases}$$

with $n \geqslant 3$ (and ϕ having a positive lower bound). We find some conditions on the initial value ϕ such that the local solutions of (P) vanish in finite time. Meanwhile, we obtain optimal conditions on ϕ for global existence and study the large time behavior of those global solutions. In particular, we prove that if $\nu > 0$ and $n \geqslant 3$,

$$\phi(x) \geqslant \gamma u_s(x) = \gamma \left[\frac{2}{\nu+1}\left(n - 2 + \frac{2}{\nu+1}\right)\right]^{-1/(\nu+1)} |x|^{2/(\nu+1)},$$

where u_s is a singular equilibrium of (P) and $\gamma > 1$, then (P) has a (unique) global classical solution u with $u \geqslant \gamma u_s$ and

$$u(x,t) \geqslant (\nu+1)^{1/(\nu+1)}\left(\gamma^{\nu+1} - 1\right)^{1/(\nu+1)} t^{1/(\nu+1)}.$$

On the other hand, the structure of positive radial solutions of the steady-state of (P) is studied and some interesting properties of the positive solutions are obtained. Moreover, the stability and weakly asymptotic stability of the positive radial solutions of the steady-state of (P) are also discussed.

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1. Introduction

In this paper we consider the following Cauchy problem

$$u_t = \Delta u - u^{-\nu}, \quad x \in \mathbf{R}^n,\ t > 0,\ \nu > 0,$$
$$u|_{t=0} = \phi \in C_{LB}\big(\mathbf{R}^n\big), \tag{1.1}$$

where $n \geqslant 3$,

$$C_{LB}\big(\mathbf{R}^n\big) = \Big\{\phi \in C(\mathbf{R}^n)\colon\ \phi > 0 \text{ in } \mathbf{R}^n \text{ with } \phi_{\min} = \min_{\mathbf{R}^n} \phi > 0 \text{ and}$$
$$\text{there exist } \kappa \geqslant 0 \text{ and } M > 0 \text{ such that } |x|^{-\kappa}\phi(x) \leqslant \Lambda \text{ for } |x| \geqslant M\Big\}.$$

Problem (1.1) appears in several applications in mechanics and physics, and in particular can be used to model the electrostatic Micro-Electromechanic System (MEMS) devices. See [4–8] and the references therein. In particular, in [5,6] and [7], Ghoussoub and Guo give a thorough study on the following problem

$$u_t = \Delta u - \frac{\lambda f(x)}{u^2}, \quad x \in \Omega,\ t > 0,\ \nu > 0,$$
$$u(x,0) = 1 \quad \text{for } x \in \Omega, \qquad u(x,t) = 1 \quad \text{for } x \in \partial\Omega, \tag{1.2}$$

where $\lambda > 0$, $f(x)$ is a positive function and Ω is a bounded smooth domain in $\mathbf{R}^n$.

Problem (1.1) can also be considered as a simplified second-order version for the dynamics of thin films of viscous fluids. Equations of the type

$$u_t = -\nabla \cdot \big(f(u)\nabla\Delta u\big) - \nabla \cdot \big(g(u)\nabla u\big)$$

have been used to model the dynamics of thin films of viscous fluids, where $z = u(x,t)$ is the height of the air/liquid interface. The zero set $\Sigma_u = \{u = 0\}$ is the liquid/solid interface and is sometimes called set of *ruptures*. Ruptures play a very important role in the study of thin films. The coefficient $f(u)$ reflects surface tension effects—a typical choice is $f(u) = u^3$. The coefficient of the second-order term can reflect additional forces such as gravity $g(u) = u^3$, van der Waals interactions $g(u) = u^m$, $m < 0$. For more background on thin films, we refer to [1–3,14,16–18,24–26] and the references therein. By choosing $f(u) = u^p$, $g(u) = u^{-m}$, (1.2) is equivalent to a fourth-order equation

$$u_t = -\nabla \cdot \big(u^p \nabla\big(\Delta u - \nu^{-1}u^{-\nu}\big)\big) \tag{1.3}$$

with $\nu = p + m - 1$. In general (1.3) is quite difficult to study. Equation (1.1), though simplified, has the same difficulty (i.e., the problem of *ruptures*) and the set of steady-states of (1.1) is contained in the set of steady-states of (1.3). So the study of (1.1) may be useful for that of (1.3).

The corresponding Cauchy problem

$$u_t = \Delta u + u^p, \quad x \in \mathbf{R}^n,\ t > 0,\ p > 1,$$
$$u|_{t=0} = \phi \in C_0(\mathbf{R}^n) \equiv C(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n), \quad \phi \geqslant 0,\ \phi \not\equiv 0, \tag{1.4}$$

has been studied by many authors. Various existence, blow-up, stability and instability results have been obtained for (1.4), see [11–13,15,19,23]. In this paper, we consider (1.1). Unlike (1.4), the main concern for (1.1) is when solution vanishes (i.e., *ruptures*). We will first obtain the existence of global positive solutions of (1.1) with some of the initial values ϕ. Then we study finite time vanishing behavior of the nonnegative solutions of (1.1) with other initial values ϕ. Finally, we study the structure and stability properties of positive radial solutions of the steady-state of (1.1), i.e., the following elliptic equation

$$\Delta u = u^{-\nu} \quad \text{in } \mathbf{R}^n,\ \nu > 0. \tag{1.5}$$

It is clear that problem (1.1) has a singular nonlinearity, which is not Lipschitz near $u = 0$. The usual method used by many authors to deal with the problem (1.4) cannot be directly used to deal with (1.1). On the other hand, we will see that the finite time blow-up behavior of the nonnegative solutions of (1.4) under some of the initial values ϕ cannot occur for the nonnegative solutions of (1.1). Instead, for (1.1), the finite time vanishing behavior of the nonnegative solutions will occur for some of the initial values ϕ.

We only study (1.1) with an initial value ϕ which has a positive lower bound. It will be interesting to consider the case that $\phi_{\min} = 0$. Many of the techniques in this paper are adopted from those in treating (1.4). We refer to in particular the book [12] and the paper [13].

The organization of the paper is as follows. In Section 2, we present the comparison principle for (1.1). In Section 3, we prove the existence of local solutions. In Section 4, we study the radially symmetric steady-state of (1.1). We derive some key exponents which determine the stability. In Section 5, we give (optimal) necessary conditions for global existence and finite time vanishing. Finally in Section 6, we discuss the stability and weakly stability of positive radial steady-states of (1.1).

2. Preliminaries

Suppose D is an unbounded domain in $\mathbf{R}^n$ with ∂D satisfying the exterior sphere condition. Let $T > 0$, $\Omega = D \times (0, T)$, and $\Gamma = \partial D \times (0, T) \cup \overline{D} \times \{0\}$. For a given nonnegative function $\psi \in C(\Gamma)$, we consider the following boundary value problem

$$u_t = \Delta u - u^{-\nu} \quad \text{in } \Omega, \qquad u|_\Gamma = \psi. \tag{2.1}$$

Definition 2.1. We call a nonnegative function u a *continuous weak* (*c.w.*) *super-solution* (*sub-solution*) *of* (2.1) if u is continuous on $\overline{\Omega}$, $u|_\Gamma \geqslant (\leqslant)\, \psi$ and $u_t \geqslant (\leqslant)\, \Delta u - u^{-\nu}$ in the distributional sense, i.e., for any $\eta \in C^{2,1}(D \times [0, T])$ with $\eta \geqslant 0$ and $\operatorname{supp} \eta(\cdot, t) \Subset D$ for all $t \in [0, T]$,

$$\int_D u(x,t)\eta(x,t)\,dx\Big|_{t=0}^{t=T_1}$$
$$\geqslant (\leqslant) \int_0^{T_1}\int_D \big[u(x,s)(\Delta\eta+\eta_t)(x,s)-\eta(x,s)u^{-\nu}\big]\,dx\,ds,$$

if $T_1 \in [0,T]$. If u is a c.w. super-solution and also a c.w. sub-solution of (2.1), we say u is a continuous weak (c.w.) solution. We call a function u a classical solution of (2.1) if $u \in C^{2,1}(\Omega)\cap C(\overline{\Omega})$ and (2.1) is satisfied.

The monotonicity method for the problem

$$u_t = \Delta u + f(x,t,u) \quad \text{in } \Omega, \qquad u|_\Gamma = \psi \tag{2.2}$$

when D is bounded was settled by Sattinger [22] provided that f is locally Lipschitz continuous in u uniformly for (x,t). When D is unbounded and $f(x,t,u)$ is continuous on $\overline{\Omega}\times\mathbf{R}$ and locally Lipschitz continuous in u uniformly for (x,t) in any bounded subset of Ω, the monotonicity method is derived in Lemma 1.2 of [23]. For our problem (2.1) here, it is clear that the nonlinearity is not Lipschitz for u near 0. The following lemma is a generalization of Lemma 1.2 of [23].

Lemma 2.2. *Assume that $\bar u$ and $\underline{u}$ are positive continuous weak super- and sub-solutions of* (2.1) *with $\bar u \geqslant \underline{u} \geqslant \min_\Omega \underline{u} > 0$ on $\overline{\Omega}$. Then* (2.1) *has a classical solution u satisfying $\underline{u} \leqslant u \leqslant \bar u$ on $\overline{\Omega}$.*

Proof. The proof is exactly the same as that of Lemma 1.2 of [23] because of the property of $\underline{u}$. Indeed, if we denote $f(u) = -u^{-\nu}$, we easily know that f is Lipschitz and is increasing with respect to $u \in [\min_\Omega \underline{u}, \max_\Omega \bar u]$ since $\min_\Omega \underline{u} > 0$. □

Remark. It is unclear if the conclusion of Lemma 2.2 still hold if $\min_\Omega \underline{u} = 0$.

Next, we recall a comparison principle of Phragmèn–Lindelöf type (see [23, Lemma 1.3]).

Lemma 2.3. *Suppose $\bar u$ and $\underline{u}$ are continuous weak super- and sub-solutions of the problem*

$$u_t = \Delta u + f(u) \quad \text{in } \Omega, \qquad u|_\Gamma = \psi \tag{2.3}$$

and $(\bar u - \underline{u})(x,t) \geqslant -B\exp[\beta|x|^2]$ on Ω with B and $\beta > 0$. Assume $f(\bar u(x,t)) - f(\underline{u}(x,t)) \geqslant C(x,t)(\bar u - \underline{u})(x,t)$ where $C \in C^{\alpha,\alpha/2}_{\mathrm{loc}}(\Omega)$ and $C(x,t) \leqslant C_0(|x|^2+1)$ on Ω for some $C_0 > 0$. Then $\bar u \geqslant \underline{u}$ on $\overline{\Omega}$.

3. Local solutions

In this section, we shall establish local existence of nonnegative solutions for the Cauchy problem (1.1) and some properties of local solutions are also studied. In what follows we denote $C = C(\ldots)$ positive constants, besides the arguments inside the parenthesis, which may vary line from line. We need the following

Definition 3.1. We call a function u a C_0-*mild solution of* (1.1) *on* $\mathbf{R}^n \times [0, T)$ if

(i) $u \in C(\mathbf{R}^n \times [0, T'])$ with $u_{\min} := \min_{\mathbf{R}^n \times [0,T']} u > 0$ for any $0 < T' < T$;
(ii) $u(x,t) = (e^{t\Delta}\phi - \int_0^t e^{(t-s)\Delta} u^{-\nu}(\cdot, s)\,ds)(x)$ for all $(x,t) \in \mathbf{R}^n \times [0, T)$, where

$$e^{t\Delta}\phi = (4\pi t)^{-n/2} \int\limits_{\mathbf{R}^n} \exp\left(-\frac{|x-y|^2}{4t}\right)\phi(y)\,dy. \tag{3.1}$$

We define a C_0-*mild super-solution* (*sub-solution*) by replacing "=" in (ii) by "$\geqslant$" ("$\leqslant$").

Remark. It is known from Lemma 1.5 of [23] that a positive continuous weak solution of (1.1) satisfying (i) of Definition 3.1 is also a C_0-*mild* solution. The converse of this is also true by the proof of Lemma 1.5 of [23]. By the regularity theory for parabolic equations, a C_0-*mild* solution u belongs to $C^{2,1}_{\mathrm{loc}}(\mathbf{R}^n \times (0, T))$. Also from Lemma 1.5 of [23], we have

Lemma 3.2. *If $u \in C_{LB}(\mathbf{R}^n)$ is a positive continuous weak super-solution* (*sub-solution*) *of the elliptic equation $\Delta u = u^{-\nu}$ in $\mathbf{R}^n$ ($n \geqslant 2$ and $\nu > 0$), then u is a C_0-mild super-solution* (*sub-solution*) *of* (1.1) *provided $\phi \leqslant (\geqslant)\, u$.*

Now we obtain local existence of solutions of (1.1) as well as some properties of the local solutions of (1.1).

Theorem 3.3. *Let $\phi \in C_{LB}(\mathbf{R}^n)$. Then* (1.1) *has a unique C_0-mild solution u on $\mathbf{R}^n \times [0, T_\phi)$ such that if $T_\phi < \infty$, then $\lim_{t \to T_\phi^-} \min_{\mathbf{R}^n} u(\cdot, t) = 0$. Furthermore, if ϕ is radial, then u is radial in x; if ϕ is radial and radially nondecreasing, then u is nondecreasing in $r = |x|$.*

Remark. If $T_\phi < \infty$, then u vanishes at a finite time T_ϕ. We also say that u has the behavior of finite time vanishing.

Proof of Theorem 3.3. Defining $\rho = \phi_{\min} > 0$, we first establish the local existence of (1.1). We will find $t_0 > 0$ and $\tilde{\rho} > 0$ depending upon ρ, ν and n such that (1.1) has a unique C_0-mild solution $u(x,t)$ satisfying

$$u(x,t) \geqslant \tilde{\rho} \quad \text{for } (x,t) \in \mathbf{R}^n \times [0, t_0].$$

Define

$$F(u) = e^{t\Delta}\phi - \int\limits_0^t e^{(t-s)\Delta} u^{-\nu}(\cdot, s)\,ds.$$

For $0 < \tilde{\rho} < \rho$ which will be determined below, we construct a sequence $\{u_k\}$ as follows

$$u_k(x,t) = e^{t\Delta}\phi - \int\limits_0^t e^{(t-s)\Delta} u_{k-1}^{-\nu}(y,s)\,ds \quad (k = 0, 1, 2, \ldots)$$

with $u_{-1}(x,t)=\tilde{\rho}$. Since

$$\begin{aligned} e^{t\Delta}\phi &= (4\pi t)^{-n/2}\int_{\mathbf{R}^n}\exp\left(-\frac{|x-y|^2}{4t}\right)\phi(y)\,dy \\ &\geqslant \rho(4\pi)^{-n/2}\int_{\mathbf{R}^n}\exp\left(-\frac{|\eta|^2}{4}\right)d\eta, \end{aligned}$$

if we choose $0<\tilde{\rho}<\frac{1}{2}\min\{\rho, 1, \rho(4\pi)^{-n/2}\int_{\mathbf{R}^n}\exp(-\frac{|\eta|^2}{4})\,d\eta\}$, we have that

$$e^{t\Delta}\phi \geqslant 2\tilde{\rho} \quad \text{for } (x,t)\in\mathbf{R}^n\times(0,\infty). \tag{3.2}$$

Noting that $(4\pi)^{-n/2}\int_{\mathbf{R}^n}\exp(-\frac{|\eta|^2}{4})\,d\eta=1$, we also have

$$\begin{aligned} e^{t\Delta}\phi &= (4\pi t)^{-n/2}\int_{\mathbf{R}^n}\exp\left(-\frac{|x-y|^2}{4t}\right)\phi(y)\,dy \\ &\leqslant (4\pi t)^{-n/2}\int_{\mathbf{R}^n}\exp\left(-\frac{|x-y|^2}{4t}\right)C\big(1+|y|\big)^{\kappa}\,dy \\ &= (4\pi)^{-n/2}\int_{\mathbf{R}^n}\exp\left(-\frac{|\eta|^2}{4}\right)C\big(1+|x|+t^{1/2}|\eta|\big)^{\kappa}\,d\eta \\ &\leqslant 2^{\kappa}C\big(1+|x|\big)^{\kappa}+2^{\kappa}CBt^{\kappa/2} \\ &\leqslant C\big(1+|x|\big)^{\kappa} \end{aligned}$$

if $0<t<1$, where $B=(4\pi)^{-n/2}\int_{\mathbf{R}^n}|\eta|^{\kappa}\exp(-\frac{|\eta|^2}{4})\,d\eta$.

On the other hand, we have that

$$u_0(x,t)=e^{t\Delta}\phi-\int_0^t e^{(t-s)\Delta}\tilde{\rho}^{-\nu}\,ds \geqslant 2\tilde{\rho}-\tilde{\rho}^{-\nu}At\geqslant\tilde{\rho}$$

if we choose $0<t<\tilde{\rho}^{\nu+1}$. It is clear that

$$u_0(x,t)\leqslant C\big(1+|x|\big)^{\kappa}.$$

Thus,

$$\tilde{\rho}\leqslant u_0(x,t)\leqslant C\big(1+|x|\big)^{\kappa} \quad \text{for } (x,t)\in\mathbf{R}^n\times[0,t_1), \tag{3.3}$$

where $t_1=\min\{1,\tilde{\rho}^{\nu+1}\}$. We can easily see that

$$u_k(x,t)\geqslant\tilde{\rho} \quad \text{for } (x,t)\in\mathbf{R}^n\times[0,t_1) \text{ and all } k\geqslant 0. \tag{3.4}$$

Now we use the induction method to show that

$$|u_k - u_{k-1}| \leqslant \left(\frac{2}{\tilde{\rho}^\nu}\right)^k \frac{t^k}{k!}. \tag{3.5}$$

Indeed, we know that

$$|u_1 - u_0| \leqslant \int_0^t e^{(t-s)\Delta} \left|u_0^{-\nu} - \tilde{\rho}^{-\nu}\right| ds \leqslant 2\tilde{\rho}^{-\nu} t. \tag{3.6}$$

If we assume (3.5) holds for $k = m-1$ ($\forall m > 2$) and we can show that (3.5) holds for $k = m$, we obtain that (3.5) holds for all k by the induction method. In fact,

$$\begin{aligned}
|u_m - u_{m-1}| &\leqslant \int_0^t e^{(t-s)\Delta} |u_{m-1} - u_{m-2}|\, ds \\
&\leqslant \left(\frac{2}{\tilde{\rho}^\nu}\right)^{m-1} \int_0^t e^{(t-s)\Delta} \frac{s^{m-1}}{(m-1)!}\, ds \\
&\leqslant \left(\frac{2}{\tilde{\rho}^\nu}\right)^{m-1} A \int_0^t \frac{s^{m-1}}{(m-1)!}\, ds \\
&\leqslant \left(\frac{2}{\tilde{\rho}^\nu}\right)^{m} \frac{t^m}{m!}.
\end{aligned}$$

Therefore, (3.5) holds for all $k = 1, 2, \ldots$. Choosing

$$t_0 = \min\left\{1, t_1, \tilde{\rho}^\nu\right\},$$

we have that for $0 < t \leqslant t_0$,

$$|u_k - u_{k-1}| \leqslant \frac{1}{k!} \quad \text{in } \mathbf{R}^n \times [0, t_0]. \tag{3.7}$$

Define $\zeta_k = \sum_{j=1}^k (u_j - u_{j-1})$. We easily know that $|\zeta_k| \leqslant \sum_{j=1}^k \frac{1}{j!}$ and hence $\zeta_k \to \zeta$ uniformly in $\mathbf{R}^n \times [0, t_0]$, as $k \to \infty$ and $\zeta \in C(\mathbf{R}^n \times [0, t_0])$. On the other hand, we have $u_k = u_0 + \zeta_k$ and hence $u_k \to U(x,t) := u_0 + \zeta$ uniformly in $\mathbf{R}^n \times [0, t_0]$, as $k \to \infty$ and $u_0 + \zeta \in C^0(\mathbf{R}^n \times [0, t_0])$. Moreover,

$$\tilde{\rho} \leqslant U(x,t) \leqslant C\left(1 + |x|\right)^\kappa \quad \text{in } \mathbf{R}^n \times [0, t_0]. \tag{3.8}$$

This also implies that $U(x,t)$ is a C_0-mild solution of (1.1). The uniqueness of $U(x,t)$ can be obtained by the comparison principle of Phragmèn–Lindelöf type (see Lemma 2.3). In fact, it follows from (3.8) that

$$U(x,t) \geqslant \tilde{\rho} \geqslant K\big(1+|x|\big)^{-2/(\nu+1)} \quad \text{for } (x,t) \in \mathbf{R}^n \times [0,t_0]. \tag{3.9}$$

Suppose that there are two solutions U_1, U_2 of (1.1), we have that

$$-\big[U_1^{-\nu} - U_2^{-\nu}\big] = \nu\xi^{-(\nu+1)}(U_1-U_2) = C(x,t)(U_1-U_2),$$

where $\xi = sU_1 + (1-s)U_2$ with $s \in (0,1)$. By (3.9), we easily know that $C(x,t) \leqslant C_0(1+|x|^2)$. Thus, it follows from Lemma 2.3 that $U_1 \geqslant U_2$ on $\mathbf{R}^n \times [0,t_0]$. Similarly, we can also obtain that $U_2 \geqslant U_1$ on $\mathbf{R}^n \times [0,t_0]$. This implies $U_1 \equiv U_2$ on $\mathbf{R}^n \times [0,t_0]$. The proof of the first part of this theorem can be completed now by a ladder argument.

The second part of this theorem can be obtained by arguments similar to those in the proof of Theorem 2.3 of [23]. □

Remark. It is known from the proof of Theorem 3.3 that if u is a C_0-mild solution of (1.1) in $\mathbf{R}^n \times (0,T)$, then

$$u(x,t) \leqslant C(t)\big(1+|x|\big)^{\kappa} \quad \text{for } (x,t) \in \mathbf{R}^n \times [0,T]. \tag{3.10}$$

Lemma 3.4. *Suppose that u is a positive classical solution of* (1.1) *on* $\mathbf{R}^n \times [0,T)$ *with* $u(x,t) \geqslant C(T')(1+|x|)^{-2/(\nu+1)}$ *on* $\mathbf{R}^n \times [0,T']$ *for any* $0 < T' < T$. *Then the following statements hold*:

(i) *If the initial value ϕ is radial, then u is radial in x-variable.*
(ii) *If ϕ is a continuous weak sub-solution* (*super-solution*) *but not a solution of $\Delta u = u^{-\nu}$, then $u_t(x,t) > (<)\, 0$, $t > 0$.*

Proof. (i) can be easily obtained by arguments similar to those in the proof of Lemma 2.6 of [23].

To prove (ii), we first notice from (3.10) that $u(x,t) \leqslant C(T')(1+|x|)^{\kappa}$ on $\mathbf{R}^n \times [0,T']$ for any $0 < T' \leqslant T$. Using Phragmèn–Lindelöf comparison principle (see Lemma 2.3), we have $u \geqslant \phi$. For a small $h > 0$, let $u_h(x,t) = u(x,t+h)$, $w = u_h - u$. Then $w|_{t=0} = u(\cdot,h) - \phi \geqslant 0$, $w_t - \Delta w = C(x,t)w$, where

$$C(x,t) \equiv -\frac{u_h^{-\nu} - u^{-\nu}}{u_h - u} = \nu\xi^{-(\nu+1)} \leqslant C(T')\big(1+|x|^2\big)$$

on $\mathbf{R}^n \times [0,T']$, where $\xi = su + (1-s)u_h$ with $s \in (0,1)$. By Lemma 2.3 again, $w \geqslant 0$, i.e., u is nondecreasing in t. Hence $u_t \geqslant 0$ if $t > 0$. Now (ii) follows from the strong maximum principle. Another part of (ii) can be treated similarly. □

4. The steady-state of (1.1)

In this section we study the structure of nonnegative solutions of the steady-state of (1.1):

$$\Delta u = u^{-\nu} \quad \text{in } \mathbf{R}^n,\ u \geqslant 0. \tag{4.1}$$

In what follows, we set

$$B_R = \{x \in \mathbf{R}^n;\ |x| < R\},$$

$$\delta = \frac{2}{\nu+1}, \qquad L = [\delta(n-2+\delta)]^{-1/(\nu+1)},$$

$$\nu_c = \begin{cases} \frac{n-2(n-1)^{1/2}}{2(n-1)^{1/2}-(n-4)} & \text{for } 3 \leqslant n \leqslant 9, \\ +\infty & \text{for } n \geqslant 10. \end{cases}$$

Definition 4.1. We say that u is a *regular solution of* (4.1) if $u \in C^2(\mathbf{R}^n)$ and u *satisfies* (4.1). We call u a singular solution of (4.1) if $u \in C^2(\mathbf{R}^n \setminus \{0\}) \cap C(\mathbf{R}^n)$ satisfies (4.1) in $\mathbf{R}^n \setminus \{0\}$ with nonremovable zero at $x = 0$.

Proposition 4.2. *When $\nu > 0$, all nontrivial nonnegative radial regular solutions of* (4.1) *are included in a family $\{u_\alpha\}_{\alpha>0}$ with u_α being the unique positive solution of the problem*

$$u'' + \frac{n-1}{r}u' = u^{-\nu} \quad \text{in } (0,\infty), \qquad u(0) = \alpha, \qquad u'(0) = 0. \tag{4.2}$$

u_α is increasing in r,

$$r^{-2/(\nu+1)}u_\alpha(r) \to L \quad \text{as } r \to +\infty,$$

$u_\alpha(r) = \alpha u_1(\alpha^{-(\nu+1)/2}r)$. The only radial singular solution of (4.1) *is*

$$u_s(r) = Lr^{2/(\nu+1)}.$$

Proof. This proposition can be obtained by phase plane analysis, see [9,10]. □

Proposition 4.3.

(i) *When $\nu > \nu_c$, if $U \not\equiv u$ are two singular* (*regular*) *solutions of $\Delta u = u^{-\nu}$ on $\overline{B_1}$ (B_1^c) with $r^{-\delta}U(r) \to L$, $r^{-\delta}u(r) \to L$ as $r \to 0^+$ $(r \to +\infty)$, then U oscillates around u.*
(ii) *When $\nu > \nu_c$, assume $\bar{u}$ $(\underline{u})$ is a radial regular super-solution* (*sub-solution*) *of* (4.1)*. If u_α is a positive radial regular solution of* (4.1) *such that $\bar{u} \geqslant u_\alpha$ $(\underline{u} \leqslant u_\alpha)$, then $u_\alpha \equiv \bar{u}$ $(\underline{u})$.*
(iii) *When $0 < \nu \leqslant \nu_c$, for any $\theta > (<)\, 1$, $\bar{u}$ $(\underline{u})$ and u_α as in* (ii)*, then $\bar{u}$ $(\underline{u})$ cannot stay above* (*below*) *θu_α.*

Proof. (i) Let $v(t) = U(r)/u(r)$, $t = \ln r$. Then v satisfies

$$v''(t) + \left(\frac{2ru'(r)}{u(r)} + n - 2\right)v'(t) + r^2u^{-(\nu+1)}(r)\big(v - v^{-\nu}\big)(t) = 0, \quad t \leqslant 0, \tag{4.3}$$

and $\lim_{t\to-\infty} v(t) = 1$. Since $r^{-\delta}u(r) \to L$ as $r \to 0^+$,

$$r^2u^{-(\nu+1)}(r) \to L^{-(\nu+1)} = \delta(n+\delta-2) \quad \text{as } r \to 0^+ \text{ or } t \to -\infty.$$

To find $\lim_{r\to 0^+}(2ru'(r))/u(r)$, we define $w_1(t) = u(r)/u_s(r)$, $t = \ln r$. Then $w_1(t) \to 1$ as $t \to -\infty$ and

$$w_1''(t) + c_0 w_1'(t) + L^{-(\nu+1)}\left(w_1 - w_1^{-\nu}\right) = 0 \quad \text{for } t \in (-\infty, 0], \tag{4.4}$$

where $c_0 = 2\delta + n - 2 > 0$. Multiplying (4.4) by w_1' and integrating it over $[t, 0]$, we have

$$\frac{1}{2}\left(w_1'(t)\right)^2 + c_0 \int_t^0 \left(w_1'(s)\right)^2 ds = O(1).$$

Thus, w_1' is bounded and $\int_{-\infty}^0 (w_1'(s))^2\, ds < +\infty$. From this and (4.4), w_1'' is bounded and hence $w_1'(t) \to 0$ as $t \to -\infty$. Since

$$w_1'(t) = \frac{1}{L}\left[r^{1-\frac{2}{\nu+1}} u'(r) - \frac{2}{\nu+1} r^{-\frac{2}{\nu+1}} u(r)\right] \to 0 \quad \text{as } r \to 0^+,$$

we obtain

$$\frac{2ru'(r)}{u(r)} = \frac{2r^{1-\frac{2}{\nu+1}} u'(r)}{r^{-\frac{2}{\nu+1}} u(r)} \to \frac{4}{\nu+1} \quad \text{as } r \to 0^+. \tag{4.5}$$

Let $w_2 = v - 1$. Then by (4.3) and the discussion above,

$$w_2''(t) + g_1(t) w_2'(t) + g_2(t) w_2 = 0 \quad \text{on } (-\infty, 0]$$

with $g_1(t) \to c_0$ and $g_2(t) = r^2 u^{-(\nu+1)}(r) \frac{(v - v^{-\nu})}{(v-1)} \to (\nu+1)L^{-(\nu+1)}$ as $t \to -\infty$. If there exists $t_m \to -\infty$ such that $v(t_m) = 1$, then we are done. So we assume $v(t) \neq 1$ for large $-t$ and hence $g_2(t)$ is well defined. By a direct calculation, when $\nu > \nu_c$,

$$\left(\lim_{t\to-\infty} g_1(t)\right)^2 - 4\left(\lim_{t\to-\infty} g_2(t)\right) = (2\delta + n - 2)^2 - 8(\delta + n - 2) < 0.$$

From this and Sturm-type arguments, we conclude that w_2 oscillates around 0 near $t = -\infty$ and the first case of (i) of this proposition follows. Another case of (i) can be discussed similarly.

(ii) Suppose that $\bar{u} \geqslant u_\alpha$ and $\bar{u} \not\equiv u_\alpha$. Let $v(t) = \bar{u}(r)/u_\alpha(r)$, $t = \ln r$. Then $v \geqslant 1$ and

$$v'' + \left(\frac{2ru_\alpha'(r)}{u_\alpha(r)} + n - 2\right) v'(t) + r^2 u_\alpha^{-(\nu+1)}(r)\left(v - v^{-\nu}\right)(t) \leqslant 0 \quad \text{on } (-\infty, +\infty). \tag{4.6}$$

Denote the coefficient of v' by $g_1(t)$. Exactly as in the proof of (i), we have $g_1(t) \to c_0$ as $t \to +\infty$ (recall from Proposition 4.2, $r^{-\delta} u_\alpha(r) \to L$ as $r \to +\infty$, so the argument there can go through.)

We claim $\lim_{t\to+\infty} v(t) = 1$. In fact, by (4.6) and the fact $v \geqslant 1$, $v'' + g_1(t) v' \leqslant 0$. Hence,

$$\exp\left(\int_0^t g_1(s)\,ds\right)v'(t) \leqslant \exp\left(\int_0^\tau g_1(s)\,ds\right)v'(\tau) \quad \text{if } t \geqslant \tau > 0. \tag{4.7}$$

Since $ru_\alpha'(r) \to 0$ as $r \to 0^+$, we have $g_1(t) \to n-2$ as $t \to -\infty$. It follows from the fact $v(t) \to \bar{u}(0)/u_\alpha(0)$ as $t \to -\infty$ that there exists a sequence $t_m \to -\infty$ such that $v'(t_m) \to 0$. Now in (4.7), letting $\tau = t_m \to -\infty$, we have either $v' < 0$ on $(-\infty, +\infty)$ or $v' \equiv 0$. (A priori, $v' \leqslant 0$ and if there exists t_0 such that $v'(t_0) < 0$, then by (4.7) again $v'(t) < 0$ if $t \geqslant t_0$. So, $v > 1$ and hence the strict inequality in (4.7) must be true which in turn implies that $v' < 0$ on $(-\infty, +\infty)$.) But $v' \equiv 0$ is impossible since $\bar{u} \not\equiv u_\alpha$. Suppose $\lim_{t\to+\infty} v(t) > 1$, then by (4.6) and the fact $v' < 0$, we have for a large T and some constant $c > 0$

$$v'' + g_1(t)v' \leqslant -c \quad \text{if } t \geqslant T.$$

This forces $v = 0$ at some t. This contradicts the facts that $v > 1$, $v' < 0$. Therefore, $\lim_{t\to+\infty} v(t) = 1$.

Now let $w = v - 1 > 0$. By (4.6) and the discussion above, we have

$$w'' + g_1(t)w' + g_2(t)w \leqslant 0, \qquad w' < 0 \quad \text{on } (-\infty, +\infty) \tag{4.8}$$

with $g_1(t) \to c_0$,

$$g_2(t) = r^2 u_\alpha^{-(\nu+1)}(r)\frac{(v - v^{-\nu})}{(v-1)} \to (\nu+1)L^{-(\nu+1)} \quad \text{as } t \to +\infty.$$

As before, when $\nu > \nu_c$,

$$\Bigl(\lim_{t\to+\infty} g_1(t)\Bigr)^2 - 4\Bigl(\lim_{t\to+\infty} g_2(t)\Bigr) < 0.$$

Then there exist $T > 0$, b_1 and c_1 such that $b_1^2 - 4c_1 < 0$, $g_1(t) < b_1$ and $g_2(t) > c_1$ if $t \geqslant T$. Observe that any solution of

$$W'' + b_1 W' + c_1 W = 0 \tag{4.9}$$

is oscillatory; in particular, there exist $b > a > T$ such that $W(a) = W(b) = 0$, $W > 0$ on (a, b) (and hence $W'(a) > 0 > W'(b)$). Multiplying (4.8) by W and (4.9) by w, we have

$$w''W + g_1(t)w'W + g_2(t)wW \leqslant 0 \quad \text{on } [a,b], \tag{4.10}$$

$$W''w + b_1W'w + c_1wW = 0 \quad \text{on } [a,b]. \tag{4.11}$$

Subtracting (4.11) from (4.10) yields

$$(Ww' - W'w)' + \bigl(g_1(t)w'W - b_1W'w\bigr) + \bigl(g_2(t) - c_1\bigr)wW \leqslant 0 \quad \text{on } [a,b].$$

Thus by the fact that $g_1(t) < b_1$, $g_2(t) > c_1$ and $w' < 0$, we have

$$(Ww' - W'w)' + b_1(w'W - W'w) < 0 \quad \text{on } (a, b),$$
$$e^{b_1 b}(Ww' - W'w)(b) < e^{b_1 a}(Ww' - W'w)(a).$$

This is impossible (note that $W'(a) > 0 > W'(b)$) and the first case of (ii) is proved. Another case of (ii) can be proved similarly.

(iii) We use the same v as in the proof of (ii), then $v \geqslant \theta > 1$ if $\bar{u} \geqslant \theta u_\alpha$. Hence the proof of (ii) implies $v = 0$ at some t. □

Proposition 4.4.

(i) *When $\nu > \nu_c$, the graph of $u_\alpha(r)$ oscillates around that of $u_s(r)$ for every $\alpha > 0$.*
(ii) *When $0 < \nu \leqslant \nu_c$, the graph of u_α does not intersect that of u_s (i.e., $u_\alpha(r) > u_s(r)$ for all $r \geqslant 0$) for every $\alpha > 0$. Furthermore, $u_\alpha(r)$ is increasing with respect to $\alpha > 0$.*

Proof. (i) follows from Proposition 4.2 and (i) of Proposition 4.3.

Now we prove (ii). Let $v(t) = r^{-\delta} u_\alpha(r)$, $t = \ln r$. Then

$$v''(t) + c_0 v'(t) + v\big(L^{-(\nu+1)} - v^{-(\nu+1)}\big) = 0 \quad \text{on } (-\infty, +\infty) \tag{4.12}$$

with $v > 0$ and $\lim_{t \to -\infty} v(t) = +\infty$, $\lim_{t \to +\infty} v(t) = L$. If the first conclusion of (ii) is not true, letting $t_1 = \min\{t\colon\ v(t) = L\}$, we have as in the proof of Proposition 4.3 that $v' < 0$ on $(-\infty, t_1]$ and $v'(t) \to -\infty$ as $t \to -\infty$. Let $q(v) = v'(t)$. Then

$$\frac{dq}{dv} + c_0 + \frac{v(L^{-(\nu+1)} - v^{-(\nu+1)})}{q} = 0 \quad \text{on } [L, \infty), \tag{4.13}$$

$q < 0$ on $[L, \infty)$ and $q(v) \to -\infty$ as $v \to +\infty$. Therefore in the (q, v)-plane, the graph of $q = q(v)$ intersects all lines $q = \mu(L - v)$ with $\mu > 0$. For each $\mu > 0$, denote the intersection with the smallest v coordinate by $(v_\mu, q(v_\mu))$. Then $\frac{dq}{dv}(v_\mu) \geqslant -\mu$ and

$$\begin{aligned}
\frac{dq}{dv}(v_\mu) &= -c_0 + \frac{v_\mu^{-\nu} - L^{-(\nu+1)} v_\mu}{\mu(L - v_\mu)} \\
&= -c_0 + \frac{L^{-(\nu+1)}(L - v_\mu) + \nu \bar{v}_\mu^{-(\nu+1)}(L - v_\mu)}{\mu(L - v_\mu)} \\
&< -c_0 + \frac{(\nu+1)L^{-(\nu+1)}}{\mu} \quad \text{for some } \bar{v}_\mu \in (L, v_\mu).
\end{aligned}$$

Thus,

$$-\mu < -c_0 + \frac{(\nu+1)L^{-(\nu+1)}}{\mu},$$
$$\mu^2 - c_0\mu + (\nu+1)L^{-(\nu+1)} > 0 \quad \text{for all } \mu > 0.$$

This implies that

$$c_0^2 - 4(\nu+1)L^{-(\nu+1)} < 0.$$

But when $0 < \nu \leqslant \nu_c$, we have that $c_0^2 - 4(\nu+1)L^{-(\nu+1)} \geqslant 0$. We reach a contradiction. The first part of (ii) is proved.

To prove the second part of (ii), we notice from the first part, $v(t) > L$ on $(-\infty, +\infty)$ and hence $v'(t) < 0$ on $(-\infty, +\infty)$ (this can be seen from (4.12) and a similar argument in the proof of (ii) of Proposition 4.3). Since $v'(t) = r(r^{-\delta}u_\alpha(r))'$, $(r^{-\delta}u_\alpha(r))' < 0$ if $r \neq 0$, this and the fact $u_\alpha(r) = \alpha u_1(\alpha^{-\frac{\nu+1}{2}} r)$ imply $\frac{\partial u_\alpha(r)}{\partial \alpha} > 0$ if $r > 0$ (we can use the transformation: $\alpha = \rho^{-\delta}$ here). This completes the proof of this proposition. □

For $0 < \nu \leqslant \nu_c$, i.e.,

$$\delta \geqslant \delta^c := \begin{cases} \frac{2(n-1)^{1/2}-(n-4)}{2} & \text{for } 3 \leqslant n \leqslant 9, \\ 0 & \text{for } n \geqslant 10 \end{cases} \qquad \big(\text{note that } \delta^c = 2/(\nu_c+1)\big),$$

we have that

$$(2\delta + n - 2)^2 - 8(\delta + n - 2) \geqslant 0.$$

Therefore, solutions of the equation

$$\sigma_{tt} + (2\delta + n - 2)\sigma_t + 2(\delta + n - 2)\sigma = 0 \tag{4.14}$$

can be written as linear combinations of $e^{-\lambda_1 t}$ and $e^{-\lambda_2 t}$, where

$$\lambda_1(\nu, n) := \frac{2\delta + n - 2 - [(2\delta + n - 2)^2 - 8(\delta + n - 2)]^{1/2}}{2} > 0, \tag{4.15}$$

$$\lambda_2(\nu, n) := \frac{2\delta + n - 2 + [(2\delta + n - 2)^2 - 8(\delta + n - 2)]^{1/2}}{2} > 0 \tag{4.16}$$

are the roots of

$$\lambda^2 - (2\delta + n - 2)\lambda + 2(\delta + n - 2) = 0. \tag{4.17}$$

To study the behavior of the solutions of (4.14), we consider three cases: (a) $2\delta + n - 2 - 2\lambda_1 < \lambda_1$, (b) $2\delta + n - 2 - 2\lambda_1 = \lambda_1$, (c) $2\delta + n - 2 - 2\lambda_1 > \lambda_1$.

If (a) occurs, we have

$$\big[(2\delta + n - 2)^2 - 8(\delta + n - 2)\big]^{1/2} < \frac{1}{2}(2\delta + n - 2) - \frac{1}{2}\big[(2\delta + n - 2)^2 - 8(\delta + n - 2)\big]^{1/2},$$

i.e.,

$$(2\delta + n - 2)^2 < 9(\delta + n - 2),$$

and

$$4\delta^2 + (4n - 17)\delta + (n - 2)(n - 11) < 0. \tag{4.18}$$

The equation

$$4\delta^2 + (4n - 17)\delta + (n - 2)(n - 11) = 0$$

has two roots:

$$\delta_1 = \frac{17 - 4n - 3(8n - 7)^{1/2}}{8},$$
$$\delta_2 = \frac{17 - 4n + 3(8n - 7)^{1/2}}{8}.$$

We easily know that

$$\delta_1 < \delta^c < \delta_2.$$

So, if $\delta^c < \delta < \delta_2$, then $2\delta + n - 2 - 2\lambda_1 < \lambda_1$; if $\delta = \delta_2$, then $2\delta + n - 2 - 2\lambda_1 = \lambda_1$ ((b) occurs); if $\delta > \delta_2$, then $2\delta + n - 2 - 2\lambda_1 > \lambda_1$ ((c) occurs). Thus, by arguments similar to those in (4.28) of [20] that any solution $\sigma(t)$ of (4.14) satisfies

$$\sigma(t) = \begin{cases} a_1 e^{-\lambda_1 t} + O(e^{-\lambda_2 t}) & \text{if } \delta^c < \delta < \delta_2, \\ a_1 e^{-\lambda_1 t} + O(te^{-2\lambda_2 t}) & \text{if } \delta = \delta_2, \\ a_1 e^{-\lambda_1 t} + O(e^{-2\lambda_2 t}) & \text{if } \delta > \delta_2. \end{cases} \tag{4.19}$$

For $0 < \nu \leqslant \nu_c$, i.e., $\delta \geqslant \delta^c$, it is straightforward to show that for $n \geqslant 3$ there exists a finite sequence $(\nu_c =)\nu_1(n) > \nu_2(n) > \cdots > \nu_N(n)$ such that $\lambda_2(\nu, n) = k\lambda_1(\nu, n)$ if and only if $\nu = \nu_k(n)$ where $N = [\frac{n}{2}]$ and $[a] =$ the largest integer which is smaller than a. It is not hard to see that

$$\nu_k(n) = \frac{n + 2 - z_k}{2 - n + z_k}, \quad k = 1, 2, \ldots, N,$$

where z_k is the only zero of $h(z) - k = 0$ and the function

$$h(z) = \frac{[z + (z^2 - 4z - 4(n - 2))^{1/2}]^2}{4(z + n - 2)}, \quad z \in \big[n - 2 + 2\delta^c, n + 2\big),$$

is strictly increasing in $[n - 2, +\infty)$. It is also possible to give a more explicit expression for $\nu_k(n)$. To this end we set $q = 2\delta + n - 2$. Then $\lambda_2 = k\lambda_1$ if and only if

$$k = \frac{q + (Q(q))^{1/2}}{q - (Q(q))^{1/2}}$$

which is equivalent to

$$\frac{k - 1}{k + 1} = \frac{(Q(q))^{1/2}}{q} = \left[1 - \frac{4}{q} - \frac{4(n - 2)}{q^2}\right]^{1/2}, \tag{4.20}$$

where Q is defined by

$$Q(q)=q^2-4q-4(n-2).$$

Squaring both sides of (4.20) and multiplying by q^2 we obtain

$$\left[1-\left(\frac{k-1}{k+1}\right)^2\right]q^2-4q-4(n-2)=0.$$

Now, $\nu_k(n)$ may be obtained by solving q explicitly. Incidentally, the fact that $k\leqslant N$ also follows easily from (4.20) since $q<n+2$ and then

$$\frac{k-1}{k+1}<\frac{(Q(n+2))^{1/2}}{n+2}=\frac{n-2}{n+2}.$$

Thus, $k<\frac{n}{2}$.

It follows from Proposition 4.2 that if u is a nonnegative radial solution of (4.1), then $\lim_{r\to+\infty} r^{-\delta}u(r)$ must always exist. Now we derive a more detailed asymptotic expansion of u near $+\infty$.

Theorem 4.5. *Let u be a nonnegative radial solution of* (4.1) *with $0<\nu\leqslant\nu_c$ and $\lim_{r\to+\infty} r^{-\delta}u(r)>0$. Then the following statements hold*:

(i) *For $\nu=\nu_k(n)$, $k=1,2,\dots,N$, we have $\lambda_2=k\lambda_1$ and, near $+\infty$,*

$$\begin{aligned}u(r)=Lr^{\delta}&+a_1r^{\delta-\lambda_1}+\cdots+a_{k-1}r^{\delta-(k-1)\lambda_1}\\&+a_kr^{\delta-k\lambda_1}\ln r+b_1r^{\delta-\lambda_2}+\cdots+O\bigl(r^{-(n+2-\epsilon)}\bigr).\end{aligned}\tag{4.21}$$

(ii) *For $\nu_{k+1}(n)<\nu<\nu_k(n)$, $k=1,2,\dots,N$* (*with the convention that $\nu_{N+1}(n)=0$*), *we have $k\lambda_1<\lambda_2<(k+1)\lambda_1$ and, near $+\infty$,*

$$\begin{aligned}u(r)=Lr^{\delta}&+a_1r^{\delta-\lambda_1}+\cdots+a_kr^{\delta-k\lambda_1}\\&+b_1r^{\delta-\lambda_2}+cr^{\delta-(k+1)\lambda_1}+\cdots+O\bigl(r^{-(n+2-\epsilon)}\bigr).\end{aligned}\tag{4.22}$$

The constant $L=(\delta(n+\delta-2))^{-1/(\nu+1)}$ and is independent of the particular solution u. The coefficients $a_2,a_3,\dots,a_N$ are uniquely determined once a_1 is determined. Moreover, once a_1 and b_1 are determined then all the coefficients in (4.21) *and* (4.22) *are uniquely determined.*

Before giving the proof of Theorem 4.5, a few remarks are in order. First of all, Theorem 4.5 is stated in a special way with the forms of expansions (4.21) and (4.22). The expansions of u near $+\infty$ may have more general forms. In particular, it will be clear from the proof below what the missing terms in (4.21) and (4.22) are. Moreover, it will also be clear from the proof below that the expansions (4.21) and (4.22) do not have to stop at $O(r^{-(n+2-\epsilon)})$; they can go on to an arbitrarily high order.

Proof of Theorem 4.5. We start with the proof of (ii). The proof is closely related to the proof of Theorem 2.5 of [13]. First, we know from Proposition 4.2 that

$$\lim_{r\to+\infty} r^{-\delta}u(r)=L.$$

Setting $W(t)=r^{-\delta}u(r)-L$ where $t=\ln r$, we see that W satisfies the equation

$$W_{tt}+(2\delta+n-2)W_t+2(\delta+n-2)W(t)-g(W)=0 \tag{4.23}$$

in $t\geqslant t_0=\ln R$ and $g(\tau)=(\tau+L)^{-\nu}-L^{-\nu}+\nu L^{-(\nu+1)}\tau$ such that

$$g(\tau)=\frac{\nu(\nu+1)}{2}L^{-(\nu+2)}\tau^2+O\bigl(\tau^3\bigr)\quad \text{for } \tau \text{ near } 0. \tag{4.24}$$

By standard arguments it follows that

$$\begin{aligned}W(t)&=a_1e^{-\lambda_1 t}+be^{-\lambda_2 t}\\&\quad+\frac{1}{\lambda_2-\lambda_1}\int_{t_0}^{t}\bigl(e^{\lambda_2(t'-t)}-e^{\lambda_1(t'-t)}\bigr)g\bigl(W(t')\bigr)\,dt',\end{aligned} \tag{4.25}$$

where a_1, b are two constants. Notice that $-\lambda_1, -\lambda_2$ are the roots of the characteristic polynomial of the linear part of (4.23), where λ_1, λ_2 are in (4.15) and (4.16). For each positive integer $M\geqslant 2$, $g(\tau)$ admits the following expansion

$$g(\tau)=d_2\tau^2+d_3\tau^3+\cdots+d_M\tau^M+O\bigl(\tau^{M+1}\bigr) \tag{4.26}$$

near $\tau=0$, where the constants $d_2, d_3, \ldots, d_M$ depend only upon ν and n. When $k=1$ we have from (4.19) that (since $\lambda_1<\lambda_2<2\lambda_1$, $\lambda_1>\lambda_2-\lambda_1=2\delta+n-2-2\lambda_1$)

$$W(t)=a_1e^{-\lambda_1 t}+O\bigl(e^{-\lambda_2 t}\bigr)$$

near $t=\infty$ (since the case $k=1$ corresponds to the case $\delta^c<\delta<\delta_2$ there). Substituting this and (4.26) (with $M=2$) into (4.25) we obtain (using the fact that $\int_{t_0}^{t}=\int_{t_0}^{\infty}-\int_{t}^{\infty}$)

$$\begin{aligned}W(t)&=a_1e^{-\lambda_1 t}+b'e^{-\lambda_2 t}-\frac{1}{\lambda_2-\lambda_1}\int_{t}^{\infty}\bigl(e^{\lambda_2(t'-t)}-e^{\lambda_1(t'-t)}\bigr)\\&\quad\times\bigl[d_2a_1^2e^{-2\lambda_1 t'}+O\bigl(e^{-(\lambda_1+\lambda_2)t'}\bigr)\bigr]\,dt'\\&=a_1e^{-\lambda_1 t}+b_1e^{-\lambda_2 t}+a_2e^{-2\lambda_1 t}+O\bigl(e^{-(\lambda_1+\lambda_2)t}\bigr),\end{aligned}$$

where the constants a_2, b' and b_1 are defined by the equalities. Note that $a_2=d_2a_1^2c(\lambda_1,\lambda_2)$ where the constant $c(\lambda_1,\lambda_2)$ depends only upon λ_1, λ_2, thus a_2 depends only upon a_1, ν and n. Now, substituting this expansion for W and (4.26) (with $M=3$) into (4.25), by similar computation, we have

$$\begin{aligned} W(t) = {} & a_1 e^{-\lambda_1 t} + b_1 e^{-\lambda_2 t} + a_2 e^{-2\lambda_1 t} \\ & + c_{11} e^{-(\lambda_1+\lambda_2)t} + b_2 e^{-2\lambda_2 t} + a_3 e^{-3\lambda_1 t} \\ & + O\bigl(e^{-(2\lambda_1+\lambda_2)t}\bigr) \quad \text{for } t \geqslant t_0, \end{aligned}$$

where $a_2 = a_2(a_1, \nu, n)$, $a_3 = a_3(a_1, \nu, n)$, $b_2 = b_2(b_1, \nu, n)$ and $c_{11} = c_{11}(a_1, b_1, \nu, n)$. Iterating this process, after finitely many steps (with the integer M in (4.26) getting larger each time) we arrive at, for each positive integer ℓ,

$$\begin{aligned} W(t) = {} & \sum_{i=1}^{\ell+k} a_i e^{-i\lambda_1 t} + \sum_{j\in J} b_j e^{-j\lambda_2 t} \\ & + \sum_{(i,j)\in I} c_{ij} e^{-(i\lambda_1+j\lambda_2)t} + O\bigl(e^{-(\ell\lambda_1+\lambda_2)t}\bigr), \end{aligned} \tag{4.27}$$

where $k = 1$ and

$$J = \{j \in \mathbf{Z}\colon\ j \geqslant 1 \text{ and } j\lambda_2 < \ell\lambda_1 + \lambda_2\},$$
$$I = \bigl\{(i,j) \in \mathbf{Z}\times\mathbf{Z}\colon\ i \geqslant 1,\ j \geqslant 1 \text{ and } i\lambda_1 + j\lambda_2 < \ell\lambda_1 + \lambda_2\bigr\}$$

and a_i depends only upon a_1, ν, n, b_j depends only upon b_1, ν, n, and c_{ij} depend only upon a_1, b_1, ν, n. (Here $\mathbf{Z} =$ the set of all integers.) Taking ℓ large enough (e.g. $\ell > (n+2)/\lambda_1$) we obtain (4.22).

For $k > 1$, the proof of (4.22) is similar. Our starting point still is (4.19) which says that in case $k > 1$,

$$W(t) = a_1 e^{-\lambda_1 t} + O\bigl(e^{-2\lambda_1 t}\bigr) \tag{4.28}$$

near $t = +\infty$. (We can check that in this case $\lambda_2 > 2\lambda_1$. This implies that $2\delta + n - 2 - 2\lambda_1 > \lambda_1$. Indeed, $\lambda_2 = \frac{2(2\delta+n-2)-2\lambda_1}{2} > 2\lambda_1$ implies $2(2\delta + n - 2 - \lambda_1) > 4\lambda_1$ and $2\delta + n - 2 - \lambda_1 > 2\lambda_1$. Thus, $2\delta + n - 2 - 2\lambda_1 > \lambda_1$.) As before, substituting this and (4.26) into (4.25), by similar computation, we have

$$W(t) = a_1 e^{-\lambda_1 t} + a_2 e^{-2\lambda_1 t} + O\bigl(e^{-\min\{3\lambda_1, \lambda_2\}t}\bigr) \tag{4.29}$$

near $t = +\infty$, where a_2 depends only upon a_1, ν and n (but is independent of b). (Here we ought to point out that although the derivation of (4.29) is similar to that of (4.27), an additional trick that $\int_{t_0}^t = \int_0^t - \int_0^{t_0}$ is needed in handling the first part of the integral in (4.25) while the second part of that integral can be handled by $\int_{t_0}^t = \int_{t_0}^\infty - \int_t^\infty$ as before.) Substituting (4.29) and (4.26) into (4.25) and iterating this process, after $(k-1)$ steps we arrive at

$$W(t) = a_1 e^{-\lambda_1 t} + a_2 e^{-2\lambda_1 t} + \cdots + a_k e^{-k\lambda_1 t} + O\bigl(e^{-\lambda_2 t}\bigr) \tag{4.30}$$

near $t = +\infty$. Repeating this process once more, we obtain

$$W(t) = a_1 e^{-\lambda_1 t} + a_2 e^{-2\lambda_1 t} + \cdots + a_k e^{-k\lambda_1 t} + b_1 e^{-\lambda_2 t} + O\bigl(e^{-(k+1)\lambda_1 t}\bigr) \tag{4.31}$$

near $t = +\infty$. Now iterating the above process with (4.31), (4.26) and (4.25) after finitely many steps we reach (4.27) and (4.22) is thus established.

Part (i) may be proved similarly by the arguments above together with the proof of Lemmas 4.3 and 4.4 in [20]. We omit the details here. □

5. Global existence and finite time vanishing

In this section we will study the global existence and large time behavior of positive solutions of the Cauchy problem

$$\begin{aligned} &u_t = \Delta u - u^{-\nu}, \quad x \in \mathbf{R}^n,\ t > 0, \\ &u|_{t=0} = \phi \in C_{LB}\big(\mathbf{R}^n\big), \end{aligned} \tag{5.1}$$

where $\nu > 0$ and $n \geqslant 3$. We begin with a necessary condition for existence of global c.w. solutions of (5.1).

For a bounded domain Ω in $\mathbf{R}^n$, let $\lambda(\Omega)$ be the first eigenvalue of $-\Delta$ on Ω with zero boundary condition, and let ψ_Ω be the corresponding eigenfunction with $\int_\Omega \psi_\Omega = 1$. Let B_R be defined in Section 4 and $\Omega_R = \{R < |x| < 2R\}$.

Proposition 5.1. *If* (5.1) *has a* (*positive*) *global c.w. solution u with $u_t \leqslant 0$ for all $t \geqslant 0$, then there exists $C > 0$ depending upon the initial value ϕ such that for $R > 0$ sufficiently large*

$$\int_{B_R} u(x,t)\psi_{B_R}(x) \geqslant C\lambda_{B_1}^{-1/\nu}(1+R)^{-\kappa/\nu}R^{2/\nu} \tag{5.2}$$

and

$$\int_{\Omega_R} u(x,t)\psi_{\Omega_R}(x) \geqslant C\lambda_{\Omega_1}^{-1/\nu}(1+2R)^{-\kappa/\nu}R^{2/\nu}. \tag{5.3}$$

Moreover,

$$\lim_{|x|\to+\infty} \sup |x|^{-(2-\kappa)/\nu} u(x,t) \geqslant (4\Lambda\lambda_{\Omega_1})^{-1/\nu} \quad \text{for all } t \geqslant 0 \tag{5.4}$$

provided $0 \leqslant \kappa \leqslant 2$;

$$\lim_{|x|\to+\infty} \sup |x|^{-(2-\kappa)/\nu} u(x,t) \geqslant (2^\kappa\Lambda\lambda_{\Omega_1})^{-1/\nu} \quad \text{for all } t \geqslant 0 \tag{5.5}$$

provided $\kappa > 2$, where $\Lambda > 0$ is defined in the definition of $C_{LB}(\mathbf{R}^n)$.

Proof. We first prove (5.2). Multiplying ψ_{B_R} to the differential equation in (5.1) and integrating over B_R, we have by Jensen's inequality (since the function $S(s) := s^{-\nu}$ is convex for $s > 0$ and $\int_{B_R} u^{-\nu}\psi_{B_R}\,dx \leqslant \int_{B_R}(u\psi_{B_R})^{-\nu}\,dx$)

$$F_R'(t) \leqslant -\lambda_{B_R} F_R(t) - F_R^{-\nu}(t) - \int_{\partial B_R} u(x,t) \frac{\partial \psi_{B_R}}{\partial \eta}\, d\sigma, \quad t \geqslant 0, \tag{5.6}$$

where η is the outward normal vector of ∂B_R (we know that $\frac{\partial \psi_{B_R}}{\partial \eta} < 0$ on ∂B_R)

$$F_R(t) = \int_{B_R} u(x,t) \psi_{B_R}(x)\, dx.$$

If there exists $t_2 \geqslant 0$ such that

$$-\lambda_{B_R} F_R(t_2) - F_R^{-\nu}(t_2) - \int_{\partial B_R} u(x,t_2) \frac{\partial \psi_{B_R}}{\partial \eta}(x)\, d\sigma < 0,$$

then by (5.6), $F_R(t)$ ultimately decreasingly $\to 0$. On the contrary, we see that

$$\left(-\lambda_{B_R} + \nu F_R^{-(\nu+1)}(t)\right) F_R'(t) - \int_{\partial B_R} \frac{\partial u(x,t)}{\partial t} \frac{\partial \psi_{B_R}}{\partial \eta}(x)\, d\sigma < 0$$

for R sufficiently large since $\lambda_{B_R} \to 0$ as $R \to \infty$ and $u_t \leqslant 0$. Thus, the function $-\lambda_{B_R} F_R(t) - F_R^{-\nu}(t) - \int_{\partial B_R} u(x,t) \frac{\partial \psi_{B_R}}{\partial \eta}\, d\sigma$ is decreasing in t. (5.6) then implies that $F_R'(t) < -c < 0$ for $t \geqslant t_2$. Therefore, $F_R(t) - F_R(t_2) \leqslant -c(t - t_2)$ and this is impossible. Since $F_R(t) \to 0$ as $t \to \infty$, there exist $0 < \tilde{c} < 1$ and $t_3 > 0$ such that

$$F_R'(t) \leqslant -c F_R^{-\nu}(t) \quad \text{if } t \geqslant t_3,$$

hence

$$\int_{F_R(t_3)}^{F_R(t)} F^{\nu}\, dF \leqslant -c(t - t_3) \quad \text{if } t \geqslant t_3.$$

This is impossible. Therefore, for all $t \geqslant 0$,

$$-\lambda_{B_R} F_R(t) - F_R^{-\nu}(t) - \int_{\partial B_R} u(x,t) \frac{\partial \psi_{B_R}}{\partial \eta}(x)\, d\sigma \geqslant 0.$$

This implies (note that $u_t \leqslant 0$ for all $t \geqslant 0$)

$$\begin{aligned} F_R(t) &\geqslant \left[\int_{\partial B_R} u(x,t) \left| \frac{\partial \psi_{B_R}}{\partial \eta}(x) \right| d\sigma \right]^{-1/\nu} \\ &\geqslant \left[\int_{\partial B_R} \phi(x) \left| \frac{\partial \psi_{B_R}}{\partial \eta}(x) \right| d\sigma \right]^{-1/\nu} \\ &\geqslant C(1+R)^{-\kappa/\nu} \left[\left| \psi_{B_R}'(R) \right| \omega_n R^{n-1} \right]^{-1/\nu}, \end{aligned}$$

where $\omega_n = |S^{n-1}|$. Since

$$-\left(r^{n-1}\psi'_{B_R}\right)' = \lambda_{B_R} r^{n-1}\psi_{B_R} \quad \text{in } [0, R], \qquad \psi_{B_R}(R) = 0,$$

we have that (note that $\int_{B_R} \psi_{B_R} = 1$)

$$-R^{n-1}\psi'_{B_R}(R) = \lambda_{B_R} \int_0^R r^{n-1}\psi_{B_R}(r)\,dr = \frac{\lambda_{B_R}}{\omega_n}.$$

Thus, since $\lambda_{B_R} = \lambda_{B_1} R^{-2}$,

$$\left|\psi'_{B_R}(R)\right| = \frac{\lambda_{B_1} R^{-(n+1)}}{\omega_n}.$$

Therefore,

$$F_R(t) \geqslant C\lambda_{B_1}^{-1/\nu}(1+R)^{-\kappa/\nu} R^{2/\nu} \quad \text{for all } t \geqslant 0. \tag{5.7}$$

To prove the second part, we choose $R > M > 0$, where M is the number in the definition of $C_{LB}(\mathbf{R}^n)$. Multiplying ψ_{Ω_R} to the differential equation in (5.1) and integrating over Ω_R, we have by Jensen's inequality that

$$F'_R(t) \leqslant -\lambda_{\Omega_R} F_R(t) - F_R^{-\nu}(t) - \int_{\partial\Omega_R} u(x,t)\frac{\partial\psi_{\Omega_R}}{\partial\eta}\,d\sigma, \quad t \geqslant 0, \tag{5.8}$$

where $F_R(t) = \int_{\Omega_R} u(x,t)\psi_{\Omega_R}(x)\,dx$. By arguments similar to those in the proof of the first part, we obtain that for all $t \geqslant 0$,

$$\begin{aligned}
F_R(t) &\geqslant \left[-\int_{\partial\Omega_R} u(x,t)\frac{\partial\psi_{\Omega_R}}{\partial\eta}\,d\sigma\right]^{-1/\nu} \\
&= \left[\int_{\partial B_{2R}} u(x,t)\left|\psi'_{\Omega_R}(2R)\right|d\sigma + \int_{\partial B_R} u(x,t)\psi'_{\Omega_R}(R)\,d\sigma\right]^{-1/\nu} \\
&\geqslant \left[\int_{\partial B_{2R}} \phi(x)\left|\psi'_{\Omega_R}(2R)\right|d\sigma + \int_{\partial B_R} \phi(x)\psi'_{\Omega_R}(R)\,d\sigma\right]^{-1/\nu} \\
&\geqslant \left[\Lambda(2R)^\kappa(2R)^{n-1}\left|\psi'_{\Omega_R}(2R)\right|\omega_n + \Lambda R^\kappa R^{n-1}\omega_n\psi'_{\Omega_R}(R)\right]^{-1/\nu} \\
&\geqslant \left[\Lambda(2R)^\kappa\omega_n\left((2R)^{n-1}\left|\psi'_{\Omega_R}(2R)\right| + R^{n-1}\psi'_{\Omega_R}(R)\right)\right]^{-1/\nu}.
\end{aligned}$$

On the other hand, we know from the equation of ψ_{Ω_R} and the fact $\int_{\Omega_R}\psi_{\Omega_R}=1$ that

$$(2R)^{n-1}\left|\psi'_{\Omega_R}(2R)\right|+R^{n-1}\psi'_{\Omega_R}(R)=\frac{\lambda_{\Omega_R}}{\omega_n}.$$

Thus,

$$F_R(t)\geqslant(\Lambda\lambda_{\Omega_1})^{-1/\nu}2^{-\kappa/\nu}R^{(2-\kappa)/\nu}.$$

Note that $\lambda_{\Omega_R}=R^{-2}\lambda_{\Omega_1}$. Moreover, we also know that if $0\leqslant\kappa\leqslant 2$

$$\begin{aligned}\sup_{R\leqslant|x|\leqslant 2R}|x|^{-(2-\kappa)/\nu}u(x,t)&\geqslant\int_{\Omega_R}|x|^{-(2-\kappa)/\nu}u(x,t)\psi_{\Omega_R}(x)\,dx\\&\geqslant(2R)^{-(2-\kappa)/\nu}\int_{\Omega_R}u(x,t)\psi_{\Omega_R}(x)\,dx\\&\geqslant(4\Lambda\lambda_{\Omega_1})^{-1/\nu}\end{aligned}$$

and hence

$$\lim_{|x|\to+\infty}\sup|x|^{-(2-\kappa)/\nu}u(x,t)\geqslant(4\Lambda\lambda_{\Omega_1})^{-1/\nu}.$$

If $\kappa>2$, we have

$$\begin{aligned}\sup_{R\leqslant|x|\leqslant 2R}|x|^{-(2-\kappa)/\nu}u(x,t)&\geqslant\int_{\Omega_R}|x|^{-(2-\kappa)/\nu}u(x,t)\psi_{\Omega_R}(x)\,dx\\&\geqslant R^{-(2-\kappa)/\nu}\int_{\Omega_R}u(x,t)\psi_{\Omega_R}(x)\,dx\\&\geqslant\left(2^{\kappa}\Lambda\lambda_{\Omega_1}\right)^{-1/\nu}\end{aligned}$$

and hence

$$\lim_{|x|\to+\infty}\sup|x|^{-(2-\kappa)/\nu}u(x,t)\geqslant\left(2^{\kappa}\Lambda\lambda_{\Omega_1}\right)^{-1/\nu}.$$

Now, combining the proof of the second part and the first part, we obtain (5.3). This completes the proof of Proposition 5.1. □

Remark. It is interesting to see that we obtain the optimal necessary conditions for the existence of global c.w. solutions of (5.1).

Corollary 5.2. *Let u be the global solution of* (5.1) *as mentioned in Proposition* 5.1. *Denote* $\lim_{t\to+\infty}u(x,t)$ *by* $u_\infty(x)$, *then the conclusion of Proposition* 5.1 *is true for* u_∞. *In particular,* $u_\infty^{-\nu}\in L^1_{\mathrm{loc}}(\mathbf{R}^n)$.

Proof. This corollary follows from the proof of Proposition 5.1 and Fatou's Lemma. Indeed, let $\tau > 0$ and $\phi_1 \in C_0^\infty(\mathbf{R}^N)$, then

$$\int\limits_{\mathbf{R}^n} u(x, s+\tau)\phi_1\,dx\Big|_{s=0}^{s=1}$$
$$= \int\limits_0^1 ds \int\limits_{\mathbf{R}^n} \big[u(x, s+\tau)\Delta\phi_1(x) - u^{-\nu}(x, s+\tau)\phi_1(x)\big]\,dx.$$

Taking nonnegative ϕ_1, from the fact that $u_\infty \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ (since $u_\infty \in C(\mathbf{R}^n)$) and Fatou's Lemma, one sees that $u_\infty^{-\nu} \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$. □

Now we prove the following theorem.

Theorem 5.3. *Suppose that $\nu > 0$, $\psi \in C_{LB}(\mathbf{R}^n)$ is a positive radial continuous weak subsolution of* (4.1) *and the initial value $\phi \geqslant \psi$ in* (5.1). *Then* (5.1) *has a unique global classical solution u satisfying*

$$u(x,t) \geqslant \psi(x) \geqslant C\big(1+|x|\big)^{2/(\nu+1)} \quad \text{on } \mathbf{R}^n \times [0,\infty). \tag{5.9}$$

Furthermore, if $\nu > \nu_c$ and ψ is not an equilibrium of (4.1)*, then* $\lim_{t\to+\infty} u(\cdot,t) = +\infty$. *This is also true if $\phi \geqslant \gamma\psi$ for some constant $\gamma > 1$ when $0 < \nu \leqslant \nu_c$ (in this case, ψ can be an equilibrium, and $u \geqslant \gamma\psi$).*

Before we prove Theorem 5.3, we give the following lemma.

Lemma 5.4. *Suppose ψ is as stated in Theorem* 5.3*, then ψ is nondecreasing in r and*

$$\psi(r) \geqslant \left(\frac{\nu+1}{2n}\right)^{1/(\nu+1)} r^{2/(\nu+1)}.$$

Proof. Let j be the standard mollifier in $\mathbf{R}^n$, and for each $\epsilon > 0$, let

$$j_\epsilon(x) = \frac{1}{\epsilon^n} j(x/\epsilon), \quad \psi_\epsilon = j_\epsilon * \psi, \quad \text{and} \quad f_\epsilon = j_\epsilon * \psi^{-\nu}.$$

Then $\Delta\psi_\epsilon \geqslant f_\epsilon$ holds classically in $\mathbf{R}^n$. Since j is radial, by Lemma 1.4 of [23], ψ_ϵ and f_ϵ are also radial. Therefore,

$$\big(r^{n-1}\psi_\epsilon'(r)\big)' \geqslant r^{n-1} f_\epsilon(r).$$

Integrating from 0 to r gives

$$r^{n-1}\psi_\epsilon'(r) \geqslant \int\limits_0^r s^{n-1} f_\epsilon(s)\,ds.$$

So, $\psi_\epsilon'(r) > 0$ $(r > 0)$ and

$$\int_0^r \frac{\psi_\epsilon'(s)}{\psi_\epsilon^{-\nu}(s)}\,ds \geqslant \int_0^r dt \int_0^t \left(\frac{s}{t}\right)^{n-1} \frac{f_\epsilon(s)}{\psi_\epsilon^{-\nu}(t)}\,ds.$$

Hence

$$\frac{1}{\nu+1}\big(\psi_\epsilon^{\nu+1}(r) - \psi_\epsilon^{\nu+1}(0)\big) \geqslant \int_0^r dt \int_0^t \left(\frac{s}{t}\right)^{n-1} \frac{f_\epsilon(s)}{\psi_\epsilon^{-\nu}(t)}\,ds.$$

Note $\psi_\epsilon \to \psi$ pointwise and $f_\epsilon(r) \to \psi^{-\nu}(r)$ as $\epsilon \to 0^+$ if $r \neq 0$. So by the Fatou Lemma,

$$\frac{1}{\nu+1}\big(\psi^{\nu+1}(r) - \psi^{\nu+1}(0)\big) \geqslant \int_0^r dt \int_0^t \left(\frac{s}{t}\right)^{n-1} \frac{\psi^{-\nu}(s)}{\psi^{-\nu}(t)}\,ds$$

and since ψ is nondecreasing, we have (since $\psi(s) \leqslant \psi(t)$, $\psi^{-\nu}(s) \geqslant \psi^{-\nu}(t)$)

$$\frac{1}{\nu+1}\psi^{\nu+1}(r) \geqslant \int_0^r dt \int_0^t \left(\frac{s}{t}\right)^{n-1} ds,$$

$$\psi(r) \geqslant \left(\frac{\nu+1}{2n}\right)^{1/(\nu+1)} r^{2/(\nu+1)}.$$

This completes the proof of this lemma. □

Proof of Theorem 5.3. The uniqueness of the global solution is a simple consequence of Theorem 3.3 and Lemma 2.3 if it exists. To prove the remaining part, consider

$$v_t = \Delta v - v^{-\nu}, \qquad v|_{t=0} = \psi. \tag{5.10}$$

Claim 1. (5.10) has a global positive classical solution v, satisfying that v is radial in x and v is nondecreasing in $t \geqslant 0$.

The proof of this claim is as follows. We know that ψ is a sub-solution of (5.10) and $e^{t\Delta}\psi = (4\pi t)^{-n/2}\int_{\mathbf{R}^n}\exp(-\frac{|x-y|^2}{4t})\psi(y)\,dy$ is a super-solution of (5.10). Using Phragmèn–Lindelöf comparison principle (see Lemma 2.3), we have $e^{t\Delta}\psi \geqslant \psi$ for all $t \geqslant 0$. Thus, applying Lemma 2.2, we have that (5.10) has a positive global solution $\psi \leqslant v \leqslant e^{t\Delta}\psi$. This together with Lemma 5.4 imply that global existence part of Theorem 5.3. By Lemma 3.4, v is radial in x, $v_t(\cdot,t) \geqslant 0$ for $0 < t < +\infty$. The proof of Claim 1 is completed.

Now we turn to the large time behavior of u. By Claim 1, $v_\infty(x) = \lim_{t\to+\infty} v(x,t)$ exists (maybe $+\infty$), v_∞ is radial and $v_\infty \geqslant \psi$.

Claim 2. If $v_\infty \not\equiv +\infty$, then v_∞ is a (radial) regular solution of (4.1).

First we notice that for any $t > 0$, $v(\cdot,t)$ is a regular sub-solution of (4.1) which is radial in x. Thus, we can easily see that $v(r,t)$ is nondecreasing about $r > 0$ for any $t > 0$. This also implies that $v_\infty(r)$ is nondecreasing in $r>0$ if $v_\infty \not\equiv +\infty$. Since v is a continuous weak solution of (5.10), we have for any $\tau>0$ and $\phi_1 \in C_0^\infty(\mathbf{R}^n)$,

$$\int_{\mathbf{R}^n} v(x,s+\tau)\phi_1(x)\,dx\bigg|_{s=0}^{s=1} = \int_0^1 ds \int_{\mathbf{R}^n} \big[v(x,s+\tau)\Delta\phi_1(x) - v^{-\nu}(x,s+\tau)\phi_1(x)\big]\,dx.$$

Let $\tau \to +\infty$, by the Lebesgue Dominated Convergence Theorem,

$$0 = \int_{\mathbf{R}^n} \big[v_\infty \Delta\phi_1 - v_\infty^{-\nu}\phi_1\big]\,dx.$$

Thus v_∞ is a distributional solution of (4.1). Now we show that $v_\infty \in L^\infty_{\mathrm{loc}}(0,+\infty)$ if $v_\infty \not\equiv +\infty$. Since $v_\infty \not\equiv +\infty$ and $v_\infty(r)$ is nondecreasing in $r>0$, we see that there exists a maximal $r_* \in (0,+\infty]$ such that $0 < v_\infty(r) \neq +\infty$ for $r \in [0,r_*)$. By the regularity theory of elliptic equations, we see that $C^2[0,r_*)$. To prove the fact $v_\infty \in L^\infty_{\mathrm{loc}}(0,+\infty)$, we only need to show that $r_* = +\infty$. On the contrary, we see that $0 < r_* < +\infty$ and $\lim_{r\to r_*^-} v_\infty(r) = +\infty$. Since v_∞ satisfies the equation

$$\big(r^{n-1}v_\infty'\big)' = r^{n-1}v_\infty^{-\nu} \quad \text{for } r \in (0,r_*),$$

we see that for $r \in (0,r_*)$,

$$0 \leqslant v_\infty'(r) = r^{1-n}\int_0^r \xi^{n-1}v_\infty^{-\nu}(\xi)\,d\xi \leqslant \frac{v_\infty^{-\nu}(0)r_*}{n}.$$

This implies that for $r \in (0,r_*)$,

$$v_\infty(r) - v_\infty(0) = \int_0^r v_\infty'(\xi)\,d\xi \leqslant \frac{v_\infty^{-\nu}(0)r_*^2}{n}$$

and

$$v_\infty(r) \leqslant \frac{v_\infty^{-\nu}(0)r_*^2}{n} + v_\infty(0).$$

This contradicts the fact that $\lim_{r\to r_*^-} v_\infty(r) = +\infty$. Now Claim 2 follows from the regularity theory for elliptic equations.

Claim 3. The function $v_\infty \equiv +\infty$ if $\nu > \nu_c$.

If ψ is a regular sub-solution of (4.1), by Claim 2, the fact $v_\infty \geqslant \psi$ and (ii) of Proposition 4.3, either $v_\infty = \lim_{\alpha\to+\infty} u_\alpha(r) \equiv +\infty$ or $v_\infty \equiv \psi$. By assumption $v_\infty \not\equiv \psi$ (ψ is not an equilibrium). So $v_\infty \equiv +\infty$. Now if ψ is not regular, observe that since v is nondecreasing in t, we can prove easily that for each $t > 0$, $v(\cdot, t)$ is a continuous weak sub-solution of (4.1). By regularity theory, $v(\cdot, t)$ is regular if $t > 0$. Note also that $v(\cdot, t)$ is radial and $v_\infty \geqslant v(\cdot, t)$. Now by (ii) of Proposition 4.3, either $v_\infty \equiv +\infty$ or $v_\infty \equiv v(\cdot, t)$ for all $t > 0$ (here we should also use the fact that $v(\cdot, t)$ is nondecreasing in t). Since the latter implies $v_\infty \equiv \psi$ which contradicts our assumption, $v_\infty \equiv +\infty$.

Claim 4. To prove the large time behavior of u when $0 < \nu \leqslant \nu_c$ and $\phi \geqslant \gamma\psi$ for some $\gamma > 1$, we follow the same line of reasoning. First replace ψ in (5.10) by $\gamma\psi$ and denote the corresponding solution of (5.10) by v^γ. Since $\gamma\psi$ is also a c.w. sub-solution of (5.10), Claim 1 is true for v^γ. Claim 2 holds for $v^\gamma_\infty \equiv \lim_{t\to+\infty} v^\gamma$ by the same argument there. To prove $v^\gamma_\infty \equiv +\infty$, noticing $\gamma\psi \leqslant v^\gamma \leqslant v^\gamma_\infty$, we have $\frac{v^\gamma_\infty}{\gamma} \geqslant \psi$. Now consider the global solution v of (5.10) (keep $v|_{t=0} = \psi$). Since $\frac{v^\gamma_\infty}{\gamma}$ is a c.w. super-solution of (5.10) ($\frac{1}{\gamma} < 1$ and v^γ_∞ is an equilibrium), we have by Phragmèn–Lindelöf comparison principle (see Lemma 2.3) that $\frac{v^\gamma_\infty}{\gamma} \geqslant v$ and hence $\frac{v^\gamma_\infty}{\gamma} \geqslant v_\infty = \lim_{t\to+\infty} v(\cdot, t)$. If $v^\gamma_\infty \not\equiv +\infty$, then v^γ_∞ and v_∞, as nontrivial regular solutions of (4.1), satisfy $\lim_{r\to+\infty} v^\gamma_\infty / v_\infty = 1$ by Proposition 4.2, a contradiction! Therefore, $v^\gamma_\infty \equiv +\infty$. This completes the proof of Theorem 5.3. □

When $\nu > \nu_c$, let $r_1(\alpha) = \min\{r \geqslant 0\colon\ u_\alpha(r) = u_s(r)\}$, $r_2(\alpha) = \min\{r > r_1(\alpha)\colon\ u_\alpha(r) = u_s(r)\}$. They are well defined by (i) of Proposition 4.4. From Proposition 4.2, we have

$$r_i(\alpha) = \alpha^{\frac{\nu+1}{2}} r_i(1), \quad i = 1, 2. \tag{5.11}$$

Proposition 5.5.

(i) *When $\nu > \nu_c$, define in $\mathbf{R}^n$*

$$\tilde{u}_\alpha(x) = \begin{cases} u_s(|x|), & |x| > r_1(\alpha), \\ u_\alpha(|x|), & |x| \leqslant r_1(\alpha), \end{cases}$$
$$\hat{u}_\alpha(x) = \begin{cases} u_s(|x|), & |x| > r_2(\alpha), \\ u_\alpha(|x|), & |x| \leqslant r_2(\alpha). \end{cases}$$

Then $\tilde{u}_\alpha$ ($\hat{u}_\alpha$) is a c.w. sub-solution (super-solution) of (4.1).

(ii) *When $\nu > 0$, for every $\alpha > 0$, $0 < \gamma \leqslant 1$, γu_α are regular super-solutions of* (4.1) *and for $\gamma > 1$, they are regular sub-solutions of* (4.1).

Proof. The proof of (ii) is trivial. We only prove (i). For all $\phi_1 \in C_0^\infty(\mathbf{R}^n)$ with $\phi_1 \geqslant 0$, we need to show

$$\int_{\mathbf{R}^n} \tilde{u}_\alpha \Delta\phi_1\, dx \geqslant \int_{\mathbf{R}^n} \tilde{u}_\alpha^{-\nu}\phi_1\, dx.$$

Let j be the standard mollifier in $\mathbf{R}^n$, and for $\epsilon > 0$, let $j_\epsilon(x) = j(x/\epsilon)/\epsilon^n$, $f(x) = \tilde{u}_\alpha^{-\nu}(x)$. Then

$$\Delta(j_\epsilon * \tilde{u}_\alpha) = j_\epsilon * f$$

classically in $B_{R-\epsilon}$ for any $R > 0$. Denote $\{|x| \leqslant r_1(\alpha)\}$ by B; then for small $\epsilon > 0$,

$$\begin{aligned}
&\int_B (j_\epsilon * \tilde{u}_\alpha)\Delta\phi_1 - \int_B (j_\epsilon * f)\phi_1 \\
&\quad = \int_B \Delta(j_\epsilon * \tilde{u}_\alpha)\phi_1 + \int_{\partial B} \frac{\partial \phi_1}{\partial \eta}(j_\epsilon * \tilde{u}_\alpha) \\
&\qquad - \int_{\partial B} \phi_1 \frac{\partial (j_\epsilon * \tilde{u}_\alpha)}{\partial \eta} - \int_B (j_\epsilon * f)\phi_1 \\
&\quad = \int_{\partial B} \left[\frac{\partial \phi_1}{\partial \eta}(j_\epsilon * \tilde{u}_\alpha) - \phi_1 \frac{\partial (j_\epsilon * \tilde{u}_\alpha)}{\partial \eta}\right]
\end{aligned}$$

(η is the outer normal vector of ∂B). Let $\epsilon \to 0^+$, then

$$\int_B [u_\alpha \Delta\phi_1 - f\phi_1] = \int_{\partial B} \left[\frac{\partial \phi_1}{\partial \eta} u_\alpha - \phi_1 \frac{\partial u_\alpha}{\partial \eta}\right]. \tag{5.12}$$

It is easy to see

$$\int_{B^c} \left[u_s \Delta\phi_1 - u_s^{-\nu}\phi_1\right] = \int_{\partial B} \left[-u_s \frac{\partial \phi_1}{\partial \eta} + \frac{\partial u_s}{\partial \eta}\phi_1\right]. \tag{5.13}$$

(5.12) and (5.13) yield

$$\int_{\mathbf{R}^n} \left[\tilde{u}_\alpha \Delta\phi_1 - \tilde{u}_\alpha^{-\nu}\phi_1\right] = \int_{\partial B} \left(\frac{\partial u_s}{\partial \eta} - \frac{\partial u_\alpha}{\partial \eta}\right)\phi_1.$$

Since $u_s'(r_1(\alpha)) > u_\alpha'(r_1(\alpha))$, the proof is finished. The proof of another claim is similar. □

We are ready to give a global existence and large time behavior result more specific than that of Theorem 5.3.

Theorem 5.6.

(i) *When $\nu > \nu_c$, if the initial value $\phi \geqslant u_s$ on $\mathbf{R}^n$, then* (5.1) *has a* (*unique*) *global classical solution u, satisfying*

$$u \geqslant u_s \quad \text{and} \quad u(\cdot, t) \to +\infty \quad \text{as } t \to +\infty.$$

(ii) *When $0 < \nu \leqslant \nu_c$, if $\phi \geqslant \gamma u_s$ for some constant $\gamma > 1$, then the conclusion of* (i) *still holds.*
(iii) *In* (i) *and* (ii), *if $\phi \geqslant \gamma u_s$ for some constant $\gamma > 1$, then $u \geqslant \gamma u_s$.*

(iv) *When* $0 < \nu \leqslant \nu_c$, *if* $\phi \geqslant \gamma u_\alpha$ *for some* $\gamma > 1$ *and some equilibrium* u_α *of* (4.1) *mentioned in Proposition* 4.2, *then the conclusion of* (i) *is true with "*$u \geqslant u_s$*" replaced by "*$u \geqslant \gamma u_\alpha$*." Furthermore, if* $\gamma = 1$, *then* (5.1) *has a* (*unique*) *global solution* $u \geqslant u_\alpha$.

Proof. We first prove (i). We shall use Proposition 5.5 to find a positive radial c.w. sub-solution of (4.1) which is below ϕ, then (i) follows from Theorem 5.3. To this end, observe that since $\phi \in C_{LB}(\mathbf{R}^n)$ and $\phi \geqslant u_s$, notice (5.11), there exists α sufficiently small such that $\phi \geqslant u_\alpha$ in $[0, r_1(\alpha)]$. We use Proposition 5.5 to find $\tilde{u}_\alpha$ desired. The proof of (i) is now completed.

To prove (ii), we look for a constant $\gamma_1 > 1$ and a radial equilibrium u_{α_1} mentioned in Proposition 4.2 such that $\phi \geqslant \gamma_1 u_{\alpha_1}$ on $\mathbf{R}^n$, then (ii) follows from (iv) which is immediate from Theorem 5.3 (note if $\gamma > 1$, γu_α is a c.w. sub-solution of (4.1)). Since $\phi \geqslant \gamma u_s$, $\liminf_{|x| \to +\infty} |x|^{-2/(\nu+1)} \phi(x) > L$. By Proposition 4.2, $L = \lim_{r \to +\infty} r^{-2/(\nu+1)} u_1(r)$. Therefore, there exist $\gamma_0 > 1$ and $R > 1$ such that

$$\phi(x) \geqslant \gamma_0 u_1(x) \quad \text{for } |x| \geqslant R. \tag{5.14}$$

Obviously, there exists $0 < \tilde{\delta} < 1$ such that $\phi(x) \geqslant \gamma_0 u_s(\tilde{\delta})$ for $0 \leqslant |x| \leqslant \tilde{\delta}$. From Proposition 4.2 again,

$$\begin{aligned} u_\alpha(r) &= \alpha u_1\big(\alpha^{-\frac{(\nu+1)}{2}} r\big) \\ &= \big(\alpha^{-\frac{(\nu+1)}{2}} r\big)^{-2/(\nu+1)} u_1\big(\alpha^{-\frac{(\nu+1)}{2}} r\big) r^{2/(\nu+1)} \\ &\to L r^{2/(\nu+1)} = u_s(r) \quad \text{as } \alpha \to 0^+. \end{aligned}$$

So there exist $1 < \gamma_2 < \gamma_0$ and $0 < \alpha_0 < 1$ such that $\gamma_0 u_s(\tilde{\delta}) > \gamma_2 u_{\alpha_0}(\tilde{\delta})$. Thus,

$$\phi(x) > \gamma_2 u_{\alpha_0}(\tilde{\delta}) \geqslant \gamma_2 u_{\alpha_0}\big(|x|\big) \quad \text{if } |x| \leqslant \tilde{\delta}. \tag{5.15}$$

Since $u_\alpha \to u_s$ uniformly on $[\tilde{\delta}, R]$ as $\alpha \to 0^+$ and $\phi > u_s$, there exists $1 < \gamma_1 < \gamma_2$ and $0 < \alpha_1 < \alpha_0$ such that

$$\phi(x) > \gamma_1 u_{\alpha_1}\big(|x|\big) \quad \text{if } \tilde{\delta} \leqslant |x| \leqslant R. \tag{5.16}$$

Combining (5.14)–(5.16) and the fact that u_α is increasing in α (see (ii) of Proposition 4.4), we have $\phi \geqslant \gamma_1 u_{\alpha_1} > \gamma_1 u_s$ on $\mathbf{R}^n$. We finish the proof of (ii).

To prove (iii), first we notice that when $\nu > \nu_c$ if we replace ϕ in the proof of (i) by ϕ/γ ($\geqslant u_s$ by the assumption), then we can find a radial c.w. sub-solution ψ of (4.1) such that $\phi/\gamma \geqslant \psi \geqslant u_s$, i.e., $\phi \geqslant \gamma\psi \geqslant \gamma u_s$. Since $\gamma\psi$ is also a c.w. sub-solution of (4.1), by Theorem 5.3, $u \geqslant \gamma\psi$ (here we should notice that the global solution of (5.1) satisfying the properties in (i) is unique by Lemma 2.3) and hence $u \geqslant \gamma u_s$. Next, when $0 < \nu \leqslant \nu_c$, by examining the proof of (ii) closely, γ_0 can be chosen arbitrarily close to γ, γ_2, γ_1 can be chosen arbitrarily close to γ_0 and γ_2, respectively. Hence γ_1 can be arbitrarily close to γ. Since $\phi \geqslant \gamma_1 u_{\alpha_1}$ and $\gamma_1 u_{\alpha_1}$ is a c.w. sub-solution of (4.1), we have $u \geqslant \gamma_1 u_{\alpha_1} (\geqslant \gamma_1 u_s)$. Letting $\gamma_1 \to \gamma$, we have $u \geqslant \gamma u_s$. (iii) is now proved. □

Next, we turn to the finite time vanishing results. The following theorem is in a direction opposite to that of Theorem 5.3.

Theorem 5.7. *Suppose that $\psi \in C_{LB}(\mathbf{R}^n)$ is a radial positive c.w. super-solution of* (4.1) *which is not a solution of* (4.1).

(i) *When $\nu > \nu_c$, if the initial value ϕ in* (5.1) $\leqslant \psi$, *then the local solution of* (5.1)*, whose existence and uniqueness are assured by Theorem* 3.3 *satisfies that $T_\phi < \infty$ and hence*

$$\lim_{t \to T_\phi^-} \min_{\mathbf{R}^n} u(\cdot, t) = 0.$$

(ii) *When $0 < \nu \leqslant \nu_c$, if the conditions on ϕ in* (i) *hold with "$\phi \leqslant \psi$" replaced by "$\phi \leqslant \gamma\psi$" for some constant $0 < \gamma < 1$, then the conclusion of* (i) *is still true.*

Proof. To prove (i), suppose contrary to the conclusion, that $T_\phi = +\infty$. Then u is a classical sub-solution of

$$v_t = \Delta v - v^{-\nu} \quad \text{in } \mathbf{R}^n \times [0, +\infty), \qquad v|_{t=0} = \psi. \tag{5.17}$$

We also know that $e^{t\Delta}\psi$ is a super-solution of (5.17), with $u(x,t) \leqslant e^{t\Delta}\psi$. The last inequality can be obtained from Lemma 2.3 (here we use the remark after the proof of Theorem 3.3). Thus, using Lemma 2.2 (note that $\min_{\mathbf{R}^n \times [0,T]} u(x,t) > 0$ for any $0 < T < \infty$), we can find a global solution $v(x,t)$ of (5.17) such that

$$v(x,t) \geqslant u(x,t) \quad \text{in } \mathbf{R}^n \times [0, +\infty).$$

This implies that $T_\psi = +\infty$. On the other hand, Lemma 3.4 implies that v is radial in x and v is nonincreasing in t. Lemma 3.4 also implies that $e^{t\Delta}\psi$ is also nonincreasing in t. Thus, $e^{t\Delta}\psi \leqslant \psi$ in $\mathbf{R}^n \times [0, +\infty)$.

Now, let $v_\infty(x) = \lim_{t \to +\infty} v(x,t)$, then v_∞ is radial and $v_\infty \leqslant v \leqslant \psi$.

Claim. The function v_∞ is a (radial) solution (either regular or singular at $x = 0$) of (4.1). To prove this, let $\tau > 0$ and $\phi_1 \in C_0^\infty(\mathbf{R}^n \setminus \{0\})$, then

$$\int_{\mathbf{R}^n} v(x, s+\tau)\phi_1(x)\,dx \bigg|_{s=0}^{s=1} = \int_0^1 ds \int_{\mathbf{R}^n} \left[v(x, s+\tau)\Delta\phi_1(x) - v^{-\nu}(x, s+\tau)\phi_1(x)\right] dx.$$

Taking nonnegative ϕ_1, from the fact that v_∞ in $C(\mathbf{R}^n)$, Corollary 5.2, one sees that $v_\infty^{-\nu} \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$. Letting $\tau \to +\infty$, by the Lebesgue Dominated Convergence Theorem,

$$0 = \int_{\mathbf{R}^n} \left[v_\infty \Delta\phi_1 - v_\infty^{-\nu}\phi_1\right] dx.$$

Taking radial ϕ_1, it is easy to see that

$$\left(r^{n-1}v_\infty'(r)\right)' - r^{n-1}v_\infty^{-\nu} = 0 \quad \text{on } (0, +\infty) \tag{5.18}$$

in the distributional sense. For sequences $\{r_m\}_{m=1}^{+\infty}$, $\{r_\ell\}_{\ell=1}^{+\infty}$ with $r_m \to 0$ as $m \to +\infty$ and $r_\ell \to 0$ as $\ell \to +\infty$ (without loss of generality, we assume $r_\ell \leqslant r_m$), we have from (5.18) that

$$\left|r_m^{n-1}v_\infty'(r_m) - r_\ell^{n-1}v_\infty'(r_\ell)\right| = \left|\int_{r_\ell}^{r_m} r^{n-1}v_\infty^{-\nu}\,dr\right| \to 0 \quad \text{as } m, \ell \to +\infty$$

since $v_\infty^{-\nu} \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$. This implies that $\lim_{r\to 0^+} r^{n-1}v_\infty'(r)$ exists. We can show that $\lim_{r\to 0^+} r^{n-1}v_\infty'(r) \geqslant 0$. Otherwise, suppose that

$$\lim_{r\to 0^+} r^{n-1}v_\infty'(r) = a < 0,$$

then there exists $\hat{r} > 0$ such that $v_\infty'(r) \leqslant (a/2)r^{1-n}$ for $0 < r < \hat{r}$. This implies that $v_\infty(0) = +\infty$, a contradiction. (Indeed, we easily see that $\lim_{r\to 0^+} r^{n-1}v_\infty'(r) = 0$.) Therefore, from (5.18), we have that $v_\infty(r)$ is nondecreasing in $r > 0$ and by a bootstrap argument, $v_\infty(r) \in C^2(0, +\infty)$. Thus v_∞ is either a regular or a singular (at $|x| = 0$) solution of (4.1). The proof of the claim is completed.

For $\nu > \nu_c$, we first show $v_\infty \not\equiv u_s$. Otherwise, $\psi \geqslant v_\infty \equiv u_s$ and hence by Theorem 5.6, $v_\infty \equiv +\infty$. A contradiction. Next, if v_∞ is a (radial) regular solution of (4.1), we still have a contradiction as follows. Since v is nonincreasing in t, it is easy to see for each $t > 0$, $v(\cdot, t)$ is a (radial) regular super-solution of (4.1) with $v(\cdot, t) \geqslant v_\infty$. By (ii) of Proposition 4.3, $v(\cdot, t) \equiv v_\infty$ for each $t > 0$ and hence $\psi = v(\cdot, t) = v_\infty$. This contradicts the assumption that ψ is not a solution of (4.1). Now the proof of (i) is completed.

To prove (ii), replace the initial value ψ in (5.17) by $\gamma\psi$. If the conclusion of (ii) is untrue, then as in the proof of (i), (5.17) has a global solution v such that v is radial in x and nonincreasing in t (note $\gamma\psi$ with $0 < \gamma < 1$ is a c.w. super-solution of (4.1)), and $v_\infty(x) = \lim_{t\to+\infty} v(x, t)$ is a radial solution (regular or singular at $x = 0$) of (4.1). If v_∞ is singular, then Proposition 4.3 implies $v_\infty \equiv u_s$ and hence $\gamma\psi \geqslant u_s$, $\psi \geqslant u_s/\gamma$. By (ii) of Theorem 5.6, the solution v_ψ of (5.17) (keep $v|_{t=0} = \psi$) tends to $+\infty$ as $t \to +\infty$ if $\psi \geqslant u_s/\gamma$. But $v_\psi \leqslant \psi$, so we reach a contradiction and hence v_∞ can only be a regular solution of (4.1). Yet this is impossible by (iv) of Theorem 5.6 and the reasoning as above. $\square$

As a consequence of Theorem 5.7 and Proposition 5.5, we have the following result which is in a direction opposite to that of Theorem 5.6.

Theorem 5.8. *The conclusion of* (i) *in Theorem* 5.7 *holds true provided that*

(i) *when* $\nu > \nu_c$, $\phi \leqslant \hat{u}_\alpha$ *for some* $\alpha > 0$,
(ii) *when* $0 < \nu \leqslant \nu_c$, $\phi \leqslant \gamma u_\alpha$ *for some* $0 < \gamma < 1$ *and some* $\alpha > 0$,
(iii) *when* $0 < \nu \leqslant \nu_c$, $\lim_{|x|\to+\infty} \sup |x|^{-2/(\nu+1)}\phi(x) < L$.

Proof. (i)–(ii) are immediate consequences of Theorem 5.7 and Proposition 5.5. To prove (iii), by (ii), it suffices to find $0<\gamma<1$ and $\alpha>0$ such that $\phi \leqslant \gamma u_\alpha$. Since

$$\lim_{|x|\to+\infty} |x|^{-2/(\nu+1)}\phi(x) < L = \lim_{r\to+\infty} r^{-2/(\nu+1)} u_1(r),$$

there exist $R>0$ and $0<\gamma<1$ such that $\phi(x) \leqslant \gamma u_1(x)$ if $|x| \geqslant R$. By the fact that $\phi > 0$ and $u_\alpha(r) = \alpha u_1(\alpha^{-(\nu+1)/2} r) \geqslant \alpha$, there exists $\alpha > 1$ such that $\phi(x) \leqslant \gamma u_\alpha(x)$ if $|x| \leqslant R$. Since u_α is increasing in α, we then have $\phi(x) \leqslant \gamma u_\alpha$ on $\mathbf{R}^n$. This completes the proof of Theorem 5.8. □

Combining Proposition 5.5, Theorems 5.6–5.8, we obtain the following theorem.

Theorem 5.9. *Suppose that $\nu > \nu_c$. Then the following conclusions hold.*

(i) *If $\phi \geqslant u_\alpha$ and $\phi \not\equiv u_\alpha$ for some $\alpha > 0$, then* (5.1) *has a unique global solution $u(x,t)$ satisfying $u(\cdot,t) \to +\infty$ as $t \to +\infty$.*
(ii) *If $\phi \leqslant u_\alpha$ and $\phi \not\equiv u_\alpha$ for some $\alpha > 0$, then the solution $u(x,t)$ of* (5.1) *must vanish in finite time.*

An important step in proving Theorem 5.9 lies in the study of the first intersection points of nearby radial solutions of the elliptic equation (4.1). We set $Z(\alpha,\beta)$ to be the first zero of $u_\alpha - u_\beta$ where $\alpha > \beta > 0$. Then $Z(\alpha,\beta) < \infty$ for all $\alpha > \beta > 0$ where $\nu > \nu_c$ by (i) of Proposition 4.3. Moreover, $Z(\alpha,\beta)$ has the following monotonicity property.

Lemma 5.10. *Assume that $\nu > \nu_c$. Then for every fixed $\alpha > 0$, we have*

$$\min\bigl\{Z(\alpha,\beta), Z(\alpha,\gamma)\bigr\} > Z(\beta,\gamma)$$

for $\alpha > \beta > \gamma > 0$.

Proof. Setting $z_1 = u_\alpha - u_\beta$ we have $z > 0$ in $[0, Z(\alpha,\beta))$ and $\Delta z_1 + k_1(x) z_1 = 0$ where

$$k_1(x) \equiv -\frac{u_\alpha^{-\nu} - u_\beta^{-\nu}}{u_\alpha - u_\beta} < \nu u_\beta^{-(\nu+1)} \quad \text{in } |x| < Z(\alpha,\beta).$$

Next, setting $z_2 = u_\beta - u_\gamma$, we have similarly that $z_2 > 0$ in $[0, Z(\beta,\gamma))$ and $\Delta z_2 + k_2(x) z_2 = 0$ where

$$k_2(x) \equiv -\frac{u_\beta^{-\nu} - u_\gamma^{-\nu}}{u_\beta - u_\gamma} > \nu u_\beta^{-(\nu+1)} \quad \text{in } |x| < Z(\beta,\gamma).$$

Suppose for contradiction that $Z(\beta,\gamma) \geqslant Z(\alpha,\beta)$. Using Lemma 2.20 of [13] (with $k(x) = \nu u_\beta^{-(\nu+1)}(x)$ and $R = Z(\alpha,\beta)$ there, it is clear that z_1 is a sub-solution of $\Delta z + k(x) z = 0$, $0 < |x| < Z(\alpha,\beta)$ and z_2 is a super-solution of this equation), we see from (2.22) of [13] that $z_1 > 0$ at $r = R = Z(\alpha,\beta)$. This contradicts the definition of $Z(\alpha,\beta)$. Therefore,

$Z(\beta,\gamma) < Z(\alpha,\beta)$, which automatically guarantees that $Z(\alpha,\gamma) > Z(\beta,\gamma)$ since $\alpha > \beta > \gamma$. This completes the proof of this lemma. □

We now begin to prove part (i) of Theorem 5.9. Without loss of generality we may assume that $\phi > u_\alpha$ in $\mathbf{R}^n$. For, the assumption that $\phi \geqslant u_\alpha$ and $\not\equiv u_\alpha$ together with the strong maximum principle for parabolic equations immediately imply that $u(x,t;\phi) > u_\alpha$ for all $x \in \mathbf{R}^n$ and $t > 0$. Thus we may replace ϕ by $u(\cdot,\epsilon;\phi)$ for some $\epsilon > 0$ if necessary.

Next, observe that by Theorem 5.3 it suffices to construct a radial continuous weak sub-solution ψ of (4.1) which is not a solution of (4.1) such that $u_\alpha \leqslant \psi \leqslant \phi$ in $\mathbf{R}^n$. To this end we first observe that $u_\beta \to u_\alpha$ uniformly in $[0, Z(\frac{3\alpha}{2},\alpha)]$ as $\beta \to \alpha$, since $Z(\frac{3\alpha}{2},\alpha) < \infty$. Thus there exists $\frac{3\alpha}{2} > \beta' > \alpha$ such that $\phi > u_{\beta'}$ in $[0, Z(\frac{3\alpha}{2},\alpha)]$. Setting

$$\psi(r) = \begin{cases} u_\alpha(r) & \text{if } r > Z(\beta',\alpha), \\ u_{\beta'}(r) & \text{if } r \leqslant Z(\beta',\alpha), \end{cases}$$

we see that $u_\alpha(x) \leqslant \psi(|x|) < \phi(x)$ for all $x \in \mathbf{R}^n$ since $Z(\frac{3\alpha}{2},\alpha) > Z(\beta',\alpha)$ by Lemma 5.10. On the other hand, it is standard to verify that ψ is a continuous weak sub-solution of (4.1). (See the proof of (i) of Proposition 5.5.) This completes the proof of part (i).

Part (ii) of this theorem may be handled in a similar fashion. As before, we may assume without loss of generality that $\phi < u_\alpha$. Since $u_\beta \to u_\alpha$ uniformly in $[0, Z(\frac{3\alpha}{2},\alpha)]$, there exists $\tilde\beta < \alpha$ such that $\phi < u_{\tilde\beta}$ in $[0, Z(\frac{3\alpha}{2},\alpha)]$. From Lemma 5.10 it follows that $Z(\alpha,\tilde\beta) < Z(\frac{3\alpha}{2},\alpha)$. Thus, setting

$$\tilde\psi(r) = \begin{cases} u_\alpha(r) & \text{if } r > Z(\alpha,\tilde\beta), \\ u_{\tilde\beta}(r) & \text{if } r \leqslant Z(\alpha,\tilde\beta), \end{cases}$$

we have $\phi(x) < \tilde\psi(|x|) \leqslant u_\alpha(x)$ for all $x \in \mathbf{R}^n$. Since $\tilde\psi$ is a continuous weak super-solution of (4.1) (see the proof of Proposition 5.5), our conclusion follows from (i) of Theorem 5.7. This completes the proof of Theorem 5.9.

6. Stability and weakly asymptotic stability results

In this section we will use the asymptotic expansions obtained in Theorem 4.5 to discuss the stability and weak asymptotic stability of the positive radial solutions of (4.1). To this end we introduce a scale of weighted norms as in [13]. For $\lambda > 0$, we define

$$\|\psi\|_\lambda = \sup_{x\in\mathbf{R}^n} \left|\left(1+|x|\right)^\lambda \psi(x)\right| \tag{6.1}$$

and

$$|||\psi|||_\lambda = \sup_{x\in\mathbf{R}^n} \left|\frac{(1+|x|)^\lambda}{\ln(2+|x|)}\psi(x)\right|, \tag{6.2}$$

where ψ is a nonnegative continuous function in $\mathbf{R}^n$. We say that a steady-state u_α of (1.1) is *stable* with respect to the norm $\|\cdot\|_\lambda$ if for every $\epsilon > 0$ there exists $\theta > 0$ such that for $\|\phi - u_\alpha\|_\lambda < \theta$ we always have $\|u(\cdot,t;\phi) - u_\alpha\|_\lambda < \epsilon$ for all $t > 0$; u_α is said to be *weakly* *asymptotically stable* with respect to $\|\cdot\|_\lambda$ if u_α is stable with respect to $\|\cdot\|_\lambda$ and there exists $\theta > 0$ such that for $\|\phi - u_\alpha\|_\lambda < \theta$ we have $\|u(\cdot, t; \phi) - u_\alpha\|_{\lambda'} \to 0$ as $t \to +\infty$ for all $\lambda' < \lambda$. Similarly we define the stability and weakly asymptotic stability with respect to the norm $|||\cdot|||_\lambda$.

Let $\delta = \frac{2}{\nu+1}$, $\lambda_1(\nu, n)$, $\lambda_2(\nu, n)$ be as in (4.15)–(4.16). By a simple calculation, we easily see that

$$\delta < \lambda_1(\nu, n) \leqslant \lambda_2(\nu, n) \tag{6.3}$$

for all $\nu > 0$ and $n \geqslant 3$.

Our main result of this section is stated as follows. (In the rest of this paper u_α is as in Proposition 4.2.)

Theorem 6.1.

(i) *If $\nu = \nu_c$ then any positive steady-state u_α of* (1.1) *is stable with respect to the norm $|||\cdot|||_{\lambda_1-\delta}$ and is weakly asymptotically stable with respect to the norm $\|\cdot\|_{\lambda_1-\delta}$.*

(ii) *For $0 < \nu < \nu_c$, any positive steady-state u_α of* (1.1) *is stable with respect to the norm $\|\cdot\|_{\lambda_1-\delta}$ and is weakly asymptotically stable with respect to the norm $\|\cdot\|_{\lambda_2-\delta}$.*

Proof. First we show that u_α is stable with respect to the norm $\|\cdot\|_{\lambda_1-\delta}$ in the case $0 < \nu < \nu_c$. From Theorem 4.5, it follows that for any $\epsilon > 0$ there exists $R_{\alpha,\epsilon}$ such that in $[R_{\alpha,\epsilon}, +\infty)$ we have

$$u_\alpha(r) = Lr^\delta + a_{1,\alpha} r^{\delta-\lambda_1} + E_\alpha(r),$$

where $|E_\alpha(r)| \leqslant \epsilon r^{\delta-\lambda_1}$ in $[R_{\alpha,\epsilon}, +\infty)$. For β near α, say, $|\beta - \alpha| < \alpha/2$, since

$$u_\beta(r) = \frac{\beta}{\alpha} u_\alpha\left(\left(\frac{\alpha}{\beta}\right)^{(\nu+1)/2} r\right),$$

we conclude that there exist constants R_ϵ and C, both independent of β, such that

$$u_\beta(r) = Lr^\delta + a_{1,\beta} r^{\delta-\lambda_1} + E_\beta(r)$$

in $[R_\epsilon, +\infty)$ where $|E_\beta(r)| \leqslant C\epsilon r^{\delta-\lambda_1}$ in $[R_\epsilon, +\infty)$ and

$$a_{1,\beta} = (\alpha/\beta)^{-[(\nu+1)\lambda_1]/2} a_{1,\alpha}.$$

Thus, in $[R_\epsilon, +\infty)$ we have

$$\left|r^{\lambda_1-\delta}(u_\beta - u_\alpha)\right| \leqslant |a_{1,\beta} - a_{1,\alpha}| + 2C\epsilon.$$

Since $|(1+r)^{\lambda_1-\delta}(u_\beta - u_\alpha)| \to 0$ uniformly in $[0, R_\epsilon]$ as $\beta \to \alpha$, it follows that

$$\limsup_{\beta\to\alpha} \|u_\beta - u_\alpha\|_{\lambda_1-\delta} \leqslant 4C\epsilon.$$

Since ϵ is arbitrary, we conclude that

$$\lim_{\beta\to\alpha}\|u_\beta-u_\alpha\|_{\lambda_1-\delta}=0.$$

This in particular implies that for any given $\epsilon>0$, there exists $\eta\in(0,\alpha/2)$ such that $\|u_{\alpha\pm\eta}-u_\alpha\|_{\lambda_1-\delta}<\epsilon$. For this η, we claim that there exists $\theta>0$ such that if $\|\phi-u_\alpha\|_{\lambda_1-\delta}<\theta$ then $u_{\alpha-\eta}\leqslant\phi\leqslant u_{\alpha+\eta}$. From this assertion and Lemma 2.2 our conclusion that u_α is stable under the norm $\|\cdot\|_{\lambda_1-\delta}$ follows immediately. We now proceed to prove this assertion. Since $0<\nu<\nu_c$, Proposition 4.4(ii) guarantees that $u_{\alpha+\eta}>u_\alpha$ and therefore $a_{1,\alpha+\eta}>a_{1,\alpha}$. Thus it follows from our arguments above that for any $\epsilon'>0$

$$\begin{aligned}r^{\lambda_1-\delta}(u_{\alpha+\eta}-u_\alpha)&=(a_{1,\alpha+\eta}-a_{1,\alpha})+\big[E_{\alpha+\eta}(r)-E_\alpha(r)\big]r^{\lambda_1-\delta}\\&\geqslant(a_{1,\alpha+\eta}-a_{1,\alpha})-2C\epsilon'\end{aligned}$$

in $[R_{\epsilon'},+\infty)$. If we choose $\theta\leqslant\frac{1}{2}(a_{1,\alpha+\eta}-a_{1,\alpha})$ and $\epsilon'<(a_{1,\alpha+\eta}-a_{1,\alpha})/(4C)$, then in $[R_{\epsilon'},+\infty)$ we have, for any $\|\phi-u_\alpha\|_{\lambda_1-\delta}<\theta$,

$$\begin{aligned}r^{\lambda_1-\delta}(u_{\alpha+\eta}-\phi)&\geqslant r^{\lambda_1-\delta}(u_{\alpha+\eta}-u_\alpha)-\big|r^{\lambda_1-\delta}(u_\alpha-\phi)\big|\\&\geqslant(a_{1,\alpha+\eta}-a_{1,\alpha})-2C\epsilon'-\theta>0.\end{aligned}$$

On the other hand, we can always choose θ even smaller if necessary so that $\phi<u_{\alpha+\eta}$ in $[0,R_{\epsilon'}]$. Hence $\phi<u_{\alpha+\eta}$ in $[0,+\infty)$. The other inequality that $\phi>u_{\alpha-\eta}$ in $[0,+\infty)$ may be derived by similar arguments, and our assertion is established.

The case $\nu=\nu_c$ can be handled in a similar fashion, we therefore omit the details.

To prove the weakly asymptotic stability of u_α, we need the following results.

Lemma 6.2. *Suppose that $\nu>0$ and h is a radial smooth function which satisfies*

$$1+h>0\quad in\ \mathbf{R}^n. \tag{6.4}$$

Then for each $\beta>0$ the problem

$$v''+\frac{n-1}{r}v'=(1+h)v^{-\nu},\qquad v(0)=\beta,\qquad v'(0)=0 \tag{6.5}$$

always has a positive solution v_β in $[0,+\infty)$.

Proof. By standard arguments one sees that (6.5) always has a unique solution v_β near $r=0$ and the solution is increasing wherever it exists. Suppose that $v_\beta(R)=+\infty$ for some $R>0$, then

$$\lim_{r\to R^-}v_\beta(r)=+\infty. \tag{6.6}$$

On the other hand, for any $r\in(0,R)$,

$$r^{n-1}v'_\beta(r)=\int_0^r\big(1+h(\xi)\big)\xi^{n-1}v^{-\nu}(\xi)\,d\xi<\beta^{-\nu}\int_0^r\big(1+h(\xi)\big)\xi^{n-1}\,d\xi.$$

This implies that $|v'_\beta(r)|$ is uniformly bounded for $r\in(0,R)$. This contradicts (6.6). $\square$

Theorem 6.3. *Suppose that $0<\nu\leqslant\nu_c$. Then for each fixed positive radial solution u_α of* (4.1) *there exist a sequence of radial strict super-solutions $\bar u_\alpha^{(1)}>\bar u_\alpha^{(2)}>\cdots>u_\alpha$ and a sequence of radial strict sub-solutions $\underline{u}_\alpha^{(1)}<\underline{u}_\alpha^{(2)}<\cdots<u_\alpha$ such that u_α is the only solution of* (4.1) *in the ordered interval $\underline{u}_\alpha^{(k)}<u_\alpha<\bar u_\alpha^{(k)}$ for every k. Moreover,*

$$\lim_{k\to+\infty}\bar u_\alpha^{(k)}=u_\alpha=\lim_{k\to+\infty}\underline{u}_\alpha^{(k)}. \tag{6.7}$$

Proof. For each $0<\nu\leqslant\nu_c$, there exists a nonnegative nontrivial smooth function h such that both h and $-h$ satisfy (6.4) with $\operatorname{supp} h\subset B_\zeta$ and $0<\zeta<1$. Denoting the solution of the problem

$$v''+\frac{n-1}{r}v'=(1\pm h)v^{-\nu}\quad\text{in }(0,+\infty),\qquad v(0)=\beta>0,\qquad v'(0)=0$$

by $v_\beta^\pm$, respectively, we see by Lemma 6.2 that both $v_\beta^\pm$ exist and are positive in $[0,+\infty)$. Note that $v_\beta^\pm$ also depend on ζ. We need to remember this in the following proofs. Obviously, v_β^- is a strict super-solution of (4.1) and v_β^+ is a strict sub-solution of (4.1). We shall use $v_\beta^\pm$ to construct the required $\bar u_\alpha^{(k)}$ and $\underline{u}_\alpha^{(k)}$. The proof is divided into the following steps.

Step 1. For every sufficiently small $\beta_1>0$, there exists $0<\alpha_1<\beta_1$ (α_1 depends on ζ) such that

$$v_{\beta_1}^->u_{\alpha_1}\quad\text{in }\mathbf{R}^n. \tag{6.8}$$

First, put $\alpha_1=\frac14\min_{B_1}v_{\beta_1}^-$. We choose $\zeta_1>0$ sufficiently small such that for $0<\zeta<\zeta_1$, $\max_{B_\zeta}u_{\alpha_1}<2\alpha_1$ and

$$v_{\beta_1}^- - u_{\alpha_1}\geqslant 4\alpha_1-2\alpha_1=2\alpha_1\geqslant u_{\alpha_1}\quad\text{in }B_\zeta,$$
$$(\nu+h)u_{\alpha_1}^{-(\nu+1)}\leqslant\nu u_{\alpha_2}^{-(\nu+1)}\quad\text{in }B_\zeta$$

for $0<\alpha_2<\alpha_1$ and sufficiently small.

Suppose that there exists $R>0$ such that $w_1=v_{\beta_1}^- - u_{\alpha_1}>0$ in B_R and $w_1(R)=0$. Then w_1 satisfies

$$\Delta w_1-\frac{(v_{\beta_1}^-)^{-\nu}-u_{\alpha_1}^{-\nu}}{v_{\beta_1}^- - u_{\alpha_1}}w_1+h\frac{(v_{\beta_1}^-)^{-\nu}}{v_{\beta_1}^- - u_{\alpha_1}}w_1=0,$$

i.e.,

$$\Delta w_1+k_1w_1=0\quad\text{in }B_R, \tag{6.9}$$

where

$$k_1 \equiv -\frac{(v_{\beta_1}^-)^{-\nu} - u_{\alpha_1}^{-\nu}}{v_{\beta_1}^- - u_{\alpha_1}} + h\frac{(v_{\beta_1}^-)^{-\nu}}{v_{\beta_1}^- - u_{\alpha_1}}.$$

Thus, for $0 \leqslant r < R$ (note that $\operatorname{supp} h \subset B_\zeta$),

$$\begin{aligned} k_1 &< \begin{cases} \nu u_{\alpha_1}^{-(\nu+1)} + h(v_{\beta_1}^-)^{-\nu} u_{\alpha_1}^{-1} & \text{if } 0 \leqslant r < \zeta, \\ \nu u_{\alpha_1}^{-(\nu+1)} & \text{if } \zeta \leqslant r < R \end{cases} \\ &\leqslant (\nu + h) u_{\alpha_1}^{-(\nu+1)} \\ &\leqslant \nu u_{\alpha_2}^{-(\nu+1)} \quad \text{with } 0 < \alpha_2 < \alpha_1. \end{aligned}$$

On the other hand, choosing $0 < \alpha_3 < \alpha_2 < \alpha_1$, if we set $w_2 = u_{\alpha_2} - u_{\alpha_3}$ then

$$\Delta w_2 + \nu u_{\alpha_2}^{-(\nu+1)} w_2 \leqslant \Delta w_2 - \frac{u_{\alpha_2}^{-\nu} - u_{\alpha_3}^{-\nu}}{u_{\alpha_2} - u_{\alpha_3}} w_2 = 0.$$

Since $w_2 > 0$ in $\mathbf{R}^n$ by Proposition 4.4(ii), which is a super-solution of

$$\Delta w + k_2 w = 0 \tag{6.10}$$

with $k_2 \equiv \nu u_{\alpha_2}^{-(\nu+1)}$ and $w_1(0) > 0$, Lemma 2.20 of [13] applies and we conclude that $w_1(R) \geqslant w_1(0) w_2(R) > 0$, a contradiction, and Step 1 is established.

Step 2. $v_\beta^+ \geqslant u_\beta$ for all $\beta > 0$.

It suffices to show that $v_\rho^+ > u_\beta$ for every $\rho > \beta$. Suppose that this is not true, i.e., there exist $\rho > \beta$ and $R > 0$ such that $w_3 = v_\rho^+ - u_\beta > 0$ in $[0, R)$ and $w_3(R) = 0$. Then w_3 satisfies

$$\Delta w_3 + k_3 w_3 \geqslant 0 \quad \text{in } B_R,$$

where $k_3 \leqslant \nu u_\beta^{-(\nu+1)}$. Choosing $\eta < \beta$ and setting $w_4 = u_\beta - u_\eta$, we have $w_4 > 0$ in $\mathbf{R}^n$ (by Proposition 4.4(ii)) and

$$\Delta w_4 + k_4 w_4 = 0 \quad \text{in } \mathbf{R}^n,$$

where

$$k_4 \equiv -\frac{u_\beta^{-\nu} - u_\eta^{-\nu}}{u_\beta - u_\eta} > \nu u_\beta^{-(\nu+1)}.$$

But then Lemma 2.20 of [13] (with $k \equiv \nu u_\beta^{-(\nu+1)}$ in (2.19) there) implies that $w_3(R) > 0$, a contradiction.

Step 3. For each α_1 in Step 1, there exists $\gamma_1 > 0$ such that

$$u_{\alpha_1} > v_\gamma^+ \quad \text{in } \mathbf{R}^n \text{ for all } 0 \leqslant \gamma \leqslant \gamma_1.$$

Set $\gamma_1 = \frac{1}{4}\min_{B_1} u_{\alpha_1}$. As in Step 1, we choose $0 < \zeta_2 < \zeta_1$ such that for $0 < \zeta < \zeta_2$, $\max_{B_\zeta} v_\gamma^+ < 2\gamma$ and

$$u_{\alpha_1} - v_\gamma^+ \geqslant v_\gamma^+ \quad \text{in } B_\zeta,$$
$$(\nu + h)u_\gamma^{-(\nu+1)} \leqslant \nu u_{\hat{\gamma}}^{-(\nu+1)} \quad \text{in } B_\zeta$$

for $0 < \hat{\gamma} < \gamma (< \gamma_1)$ sufficiently small.

Suppose that there exists $R > 0$ such that $w_5 = u_{\alpha_1} - v_\gamma^+ > 0$ in B_R where $0 < \gamma < \gamma_1$, and $w_5(R) = 0$. Then w_5 satisfies $\Delta w_5 + k_5 w_5 = 0$ in B_R where

$$k_5 \equiv -\frac{u_{\alpha_1}^{-\nu} - (v_\gamma^+)^{-\nu}}{u_{\alpha_1} - v_\gamma^+} + h\frac{(v_\gamma^+)^{-\nu}}{u_{\alpha_1} - v_\gamma^+}.$$

Thus, for $0 \leqslant r < R$,

$$\begin{aligned} k_5 &< \begin{cases} \nu(v_\gamma^+)^{-(\nu+1)} + h(v_\gamma^+)^{-(\nu+1)} & \text{if } 0 \leqslant r < \zeta, \\ \nu(v_\gamma^+)^{-(\nu+1)} & \text{if } \zeta \leqslant r < R \end{cases} \\ &\leqslant (\nu + h)(u_\gamma)^{-(\nu+1)} \quad \text{(by Step 2)} \\ &\leqslant \nu u_{\hat{\gamma}}^{-(\nu+1)} \quad \text{with } 0 < \hat{\gamma} < \gamma < \gamma_1. \end{aligned}$$

We can obtain a contradiction by the arguments similar to those in the proof of Step 1.

Step 4. For each $\alpha > 0$ there exist a radial strict super-solution $\bar{u}_\alpha^{(1)}$ of (4.1) and a radial strict sub-solution $\underline{u}_\alpha^{(1)}$ of (4.1) such that

$$\bar{u}_\alpha^{(1)} > u_\alpha > \underline{u}_\alpha^{(1)} \quad \text{in } \mathbf{R}^n. \tag{6.11}$$

Moreover, u_α is the only solution of (4.1) which satisfies (6.11).

From Steps 1 and 3 it follows that there exist small $\beta_1 > \alpha_1 > \gamma_1 > 0$ such that $v_{\beta_1}^- > u_{\alpha_1} > v_{\gamma_1}^+$ in $\mathbf{R}^n$. Now, fix β_1 and γ_1 and define

$$\alpha_1' = \sup\{\alpha \in (\gamma_1, \beta_1)\colon\ v_{\beta_1}^- > u_\alpha > v_{\gamma_1}^+ \text{ in } \mathbf{R}^n\}$$

and

$$\alpha_1'' = \inf\{\alpha \in (\gamma_1, \beta_1)\colon\ v_{\beta_1}^- > u_\alpha > v_{\gamma_1}^+ \text{ in } \mathbf{R}^n\}.$$

Obviously we have

$$v_{\beta_1}^- \geqslant u_{\alpha_1'} \geqslant u_{\alpha_1} \geqslant u_{\alpha_1''} \geqslant v_{\gamma_1}^+ \quad \text{in } \mathbf{R}^n. \tag{6.12}$$

Then, for each given $\alpha > 0$ we set

$$\bar{u}_\alpha^{(1)}(r) = \frac{\alpha}{\alpha_1'} v_{\beta_1}^-\left(\left(\frac{\alpha_1'}{\alpha}\right)^{(\nu+1)/2} r\right), \qquad \underline{u}_\alpha^{(1)}(r) = \frac{\alpha}{\alpha_1''} v_{\gamma_1}^+\left(\left(\frac{\alpha_1''}{\alpha}\right)^{(\nu+1)/2} r\right). \tag{6.13}$$

By standard scaling arguments, we have

$$\bar{u}_\alpha^{(1)}(r) \geqslant \frac{\alpha}{\alpha_1'} u_{\alpha_1'}\left(\left(\frac{\alpha_1'}{\alpha}\right)^{(\nu+1)/2} r\right) = u_\alpha(r)$$

by (6.12), and similarly $u_\alpha \geqslant \underline{u}_\alpha^{(1)}$. Since $h \geqslant 0$ and $\not\equiv 0$, $\bar{u}_\alpha^{(1)}$ and $\underline{u}_\alpha^{(1)}$ are strict super- and sub-solutions of (4.1), respectively. Hence $\bar{u}_\alpha^{(1)} > u_\alpha > \underline{u}_\alpha^{(1)}$ in $\mathbf{R}^n$ by the strong maximum principle.

It remains to show that u_α is the only solution of (4.1) which satisfies (6.11). Suppose for contradiction that there exists $\beta \neq \alpha$ such that $\bar{u}_\alpha^{(1)} > u_\beta > \underline{u}_\alpha^{(1)}$ in $\mathbf{R}^n$. Without loss of generality we may assume that $\beta > \alpha$. From $\bar{u}_\alpha^{(1)} > u_\beta$ it follows that

$$v_{\beta_1}^-(r) > \frac{\alpha_1'}{\alpha} u_\beta\left(\left(\frac{\alpha}{\alpha_1'}\right)^{(\nu+1)/2} r\right) = u_{\frac{\beta\alpha_1'}{\alpha}}(r) > u_{\alpha_1'}(r) \geqslant v_{\gamma_1}^+(r)$$

since $(\beta\alpha_1')/\alpha > \alpha_1'$. This, however, contradicts the definition of α_1'. Hence u_α is the only solution of (4.1) satisfying (6.11).

Step 5. Setting

$$\bar{h}_\alpha(r) = -h\left(\left(\frac{\alpha_1'}{\alpha}\right)^{(\nu+1)/2} r\right) \quad \text{and} \quad \underline{h}_\alpha(r) = h\left(\left(\frac{\alpha_1''}{\alpha}\right)^{(\nu+1)/2} r\right)$$

we have immediately that

$$\Delta \bar{u}_\alpha^{(1)} = (1 - \bar{h}_\alpha)\big(\bar{u}_\alpha^{(1)}\big)^{-\nu}$$

and

$$\Delta \underline{u}_\alpha^{(1)} = (1 + \underline{h}_\alpha)\big(\underline{u}_\alpha^{(1)}\big)^{-\nu} \quad \text{in } \mathbf{R}^n.$$

Now, considering the equation

$$\Delta u = \left(1 - \frac{\bar{h}_\alpha}{k}\right) u^{-\nu} \quad \text{in } \mathbf{R}^n, \tag{6.14$_k$}$$

we see that for $k = 2$, $\bar{u}_\alpha^{(1)}$ is a strict super-solution of $(6.14)_2$ and u_α is a strict sub-solution of $(6.14)_2$. Thus $(6.14)_2$ has a radial solution $\bar{u}_\alpha^{(2)}$ with $u_\alpha < \bar{u}_\alpha^{(2)} < \bar{u}_\alpha^{(1)}$ by the usual barrier method (see, e.g., the arguments used in Theorem 2.10 in [21]).

Iterating this argument, we obtain a sequence of radial strict super-solutions of (4.1) $\bar{u}_\alpha^{(1)} > \bar{u}_\alpha^{(2)} > \cdots > u_\alpha$ in $\mathbf{R}^n$. Similarly, a sequence of radial strict sub-solutions $\underline{u}_\alpha^{(1)} < \underline{u}_\alpha^{(2)} < \cdots < u_\alpha$ may be constructed by using $\underline{h}_\alpha$ and the corresponding equations

$$\Delta u = \left(1 + \frac{\underline{h}_\alpha}{k}\right) u^{-\nu} \quad \text{in } \mathbf{R}^n. \tag{6.14$'_k$}$$

Since u_α is the only solution of (4.1) satisfying (6.11), it must be the only solution of (4.1) with the property that $\bar{u}_\alpha^{(k)} > u_\alpha > \underline{u}_\alpha^{(k)}$, and our construction is complete.

Finally, we will conclude our proof by establishing (6.7). Since the sequence $\{\bar{u}_\alpha^{(k)}\colon k = 1, 2, \dots\}$ is bounded below and monotonically decreasing, from standard elliptic estimates it follows that its limit $\tilde{u}$ must be a (classical) solution of (4.1). (Note that $\bar{h}_\alpha/k \to 0$ as $k \to +\infty$ in C^2-norm.) Since $\tilde{u} \geqslant u_\alpha$, $\tilde{u}$ must also satisfy (6.11) and, $\tilde{u} \equiv u_\alpha$ by the uniqueness. Thus, $\bar{u}_\alpha^{(k)} \to u_\alpha$ as $k \to +\infty$. Similarly, $\underline{u}_\alpha^{(k)} \to u_\alpha$ as $k \to +\infty$ and (6.7) is established. This completes the proof of Theorem 6.3. □

The rest of the proof of Theorem 6.1 is still technical. In what follows, we only consider the case $(\nu_c =)\nu_1(n) > \nu > \nu_2(n)$, other cases are similar but more difficult. Where $\nu_1(n)$ and $\nu_2(n)$ are defined in Theorem 4.5.

Our next goal is to use Theorem 4.5 to obtain the asymptotic expansions of the super- and subsolutions $\bar{u}_\alpha^{(k)}$, $\underline{u}_\alpha^{(k)}$, $k = 1, 2, \dots$, obtained in Theorem 6.3 as well as the solution u_α of (4.1).

Since $\bar{u}_\alpha^{(k)}$ is a solution of $(6.14)_k$ with $1 - \frac{\bar{h}_\alpha}{k} = 1$ outside a finite ball (which is independent of k) and $\bar{u}_\alpha^{(k)} \geqslant u_\alpha$, Theorem 4.5 applies and we have

$$\bar{u}_\alpha^{(k)}(r) = Lr^\delta + \bar{a}_{1,\alpha}^{(k)} r^{\delta-\lambda_1} + \bar{b}_{1,\alpha}^{(k)} r^{\delta-\lambda_2} + \cdots + O\big(r^{-(n+2-\epsilon)}\big) \tag{6.15}$$

near $+\infty$ by (4.22). Similarly,

$$u_\alpha(r) = Lr^\delta + a_{1,\alpha} r^{\delta-\lambda_1} + b_{1,\alpha} r^{\delta-\lambda_2} + \cdots + O\big(r^{-(n+2-\epsilon)}\big) \tag{6.16}$$

near $+\infty$. For sub-solutions $\underline{u}_\alpha^{(k)}$, Theorem 4.5 still applies and

$$\underline{u}_\alpha^{(k)}(r) = Lr^\delta + \underline{a}_{1,\alpha}^{(k)} r^{\delta-\lambda_1} + \underline{b}_{1,\alpha}^{(k)} r^{\delta-\lambda_2} + \cdots + O\big(r^{-(n+2-\epsilon)}\big) \tag{6.17}$$

near $+\infty$. For, if

$$\lim_{r\to+\infty} r^{-\delta}\underline{u}_\alpha^{(k)}(r) = 0,$$

considering the solution $u_{\alpha/2}(r)$ of (4.1), since $\lim_{r\to+\infty} r^{-\delta}u_{\alpha/2}(r) = L$, we see that $u_{\alpha/2} > \underline{u}_\alpha^{(k)}$ near $+\infty$, say, in $[R, +\infty)$. Since $u_\beta \to u_\alpha$ uniformly in $[0, R]$ and $u_\alpha > \underline{u}_\alpha^{(k)}$ in $[0, R]$, there exists $\alpha/2 < \beta < \alpha$ such that $u_\beta > \underline{u}_\alpha^{(k)}$ in $[0, R]$. Since $u_\beta > u_{\alpha/2}$ in $\mathbf{R}^n$, we conclude that $u_\alpha > u_\beta > \underline{u}_\alpha^{(k)}$ in $\mathbf{R}^n$ which contradicts the uniqueness of u_α in Theorem 6.3. Thus, $\lim_{r\to+\infty} r^{-\delta}\underline{u}_\alpha^{(k)}(r) > 0$ and our Theorem 4.5 applies and gives (6.17).

It is necessary for our purposes to understand the relations between the coefficients $\bar{a}_{1,\alpha}^{(k)}$, $a_{1,\alpha}$, $\underline{a}_{1,\alpha}^{(k)}$, $\bar{b}_{1,\alpha}^{(k)}$, $b_{1,\alpha}$, $\underline{b}_{1,\alpha}^{(k)}$. Lemmas 6.4–6.6 below are essential to our weak asymptotic stability considerations.

Lemma 6.4. *For every k we have*

$$\big(\bar{a}_{1,\alpha}^{(k)} - a_{1,\alpha}\big)^2 + \big(\bar{b}_{1,\alpha}^{(k)} - b_{1,\alpha}\big)^2 > 0$$

and

$$\big(a_{1,\alpha} - \underline{a}_{1,\alpha}^{(k)}\big)^2 + \big(b_{1,\alpha} - \underline{b}_{1,\alpha}^{(k)}\big)^2 > 0.$$

Furthermore, for every $\ell \neq k$, we have

$$\big(\bar{a}_{1,\alpha}^{(k)} - \bar{a}_{1,\alpha}^{(\ell)}\big)^2 + \big(\bar{b}_{1,\alpha}^{(k)} - \bar{b}_{1,\alpha}^{(\ell)}\big)^2 > 0$$

and

$$\big(\underline{a}_{1,\alpha}^{(k)} - \underline{a}_{1,\alpha}^{(\ell)}\big)^2 + \big(\underline{b}_{1,\alpha}^{(k)} - \underline{b}_{1,\alpha}^{(\ell)}\big)^2 > 0.$$

Proof. We will only prove the first inequality since the others can be handled similarly.

Suppose for some k that $\bar{a}_{1,\alpha}^{(k)} = a_{1,\alpha}$ and $\bar{b}_{1,\alpha}^{(k)} = b_{1,\alpha}$. Then for this particular k all the coefficients in the expressions (6.15) and (6.16) are the same by Theorem 4.5. Thus,

$$\bar{u}_\alpha^{(k)} - u_\alpha(r) = O\big(r^{-(n+2-\epsilon)}\big)$$

near $+\infty$. Since $\Delta(\bar{u}_\alpha^{(k)} - u_\alpha) < 0$ in $\mathbf{R}^n$ by $(6.14)_k$ and that $\bar{u}_\alpha^{(k)} > u_\alpha$, it follows from standard arguments that

$$\big(\bar{u}_\alpha^{(k)} - u_\alpha\big)(r) \geqslant Cr^{2-n} \tag{6.18}$$

near $+\infty$ for some positive constant C (see the proof of Theorem 3.8 in [21]). This contradicts (6.18) and finishes the proof. □

Lemma 6.5. $\bar{a}_{1,\alpha}^{(k)} = a_{1,\alpha} = \underline{a}_{1,\alpha}^{(k)} > 0$ *for all k and α.*

Proof. Since $u_\alpha(r) = \alpha u_1(\alpha^{-(\nu+1)/2} r)$, we deduce from (6.16) that

$$\begin{aligned} &Lr^\delta + a_{1,\alpha} r^{\delta-\lambda_1} + b_{1,\alpha} r^{\delta-\lambda_2} + \cdots + O\big(r^{-(n+2-\epsilon)}\big) \\ &\quad = Lr^\delta + \alpha a_{1,1}\big(\alpha^{-(\nu+1)/2}\big)^{\delta-\lambda_1} r^{\delta-\lambda_1} \\ &\qquad + b_{1,1}\alpha\big(\alpha^{-(\nu+1)/2}\big)^{\delta-\lambda_2} r^{\delta-\lambda_2} + \cdots + O\big(r^{-(n+2-\epsilon)}\big). \end{aligned}$$

It then follows that

$$a_{1,\alpha} = \alpha^{(\nu+1)\lambda_1/2} a_{1,1} \quad \text{and} \quad b_{1,\alpha} = \alpha^{(\nu+1)\lambda_2/2} b_{1,1}. \tag{6.19}$$

By Proposition 4.4(ii) we conclude that u_α is increasing as α increases. This implies that $a_{1,\alpha}$ is nondecreasing in α, which in turn implies that $a_{1,1} \geqslant 0$, and $a_{1,\alpha} \geqslant 0$ by (6.19).

From (6.15)–(6.17) it follows easily that $\bar{a}_{1,\alpha}^{(k)} \geqslant a_{1,\alpha} \geqslant \underline{a}_{1,\alpha}^{(k)}$ since $\bar{u}_\alpha^{(k)} > u_\alpha > \underline{u}_\alpha^{(k)}$. If $\bar{a}_{1,\alpha}^{(k)} > a_{1,\alpha}$ then $\bar{a}_{1,\alpha}^{(k)} > a_{1,\beta}$ for all β sufficiently close to α. We then infer from (6.15) and the asymptotic expansion for u_β (with α replaced by β in (6.16)) that for every $\beta > \alpha$ and sufficiently close to α there exists $R(\beta)$ such that $\bar{u}_\alpha^{(k)} > u_\beta$ in $[R(\beta), +\infty)$. Since u_β is increasing in β, $R(\beta)$ may be chosen independent of β if β is sufficiently close to α. That is, $\bar{u}_\alpha^{(k)} > u_\beta$ in $[R, +\infty)$ for all β sufficiently close to α. On the other hand, $u_\beta \to u_\alpha$ uniformly on $[0, R]$ (since $u_\beta \to u_\alpha$ monotonically as β decreases to α and u_α is continuous), thus $\bar{u}_\alpha^{(k)} > u_\beta$ on $[0, R]$ and therefore in $\mathbf{R}^n$ for all β sufficiently close to α. This contradicts the uniqueness assertion in Theorem 6.3. Hence $\bar{a}_{1,\alpha}^{(k)} = a_{1,\alpha}$. Similarly we have $a_{1,\alpha} = \underline{a}_{1,\alpha}^{(k)}$.

It remains to show that $a_{1,\alpha} > 0$. Suppose for contradiction that $a_{1,\alpha} = 0$ for some $\alpha > 0$. Then $a_{1,\alpha} = 0$ for all α by (6.19), and therefore $\bar{a}_{1,\alpha}^{(k)} = 0 = \underline{a}_{1,\alpha}^{(k)}$ for all k by what we have just proved.

We can now repeat the arguments in the previous paragraph (which lead to the conclusion that $\bar{a}_{1,\alpha}^{(k)} = a_{1,\alpha} = \underline{a}_{1,\alpha}^{(k)}$) to conclude that $\bar{b}_{1,\alpha}^{(k)} = b_{1,\alpha} = \underline{b}_{1,\alpha}^{(k)}$ which clearly gives rise to a contradiction to Lemma 6.4. Therefore, $a_{1,\alpha} > 0$ for all $\alpha > 0$ and our proof is complete. □

Lemma 6.6. *$b_{1,\alpha}^{(k)}$ is strictly decreasing to $b_{1,\alpha}$ and $\underline{b}_{1,\alpha}^{(k)}$ is strictly increasing to $b_{1,\alpha}$ as $k \to +\infty$.*

Proof. Since $\bar{u}_{1,\alpha}^{(k)}$ is a decreasing sequence with limit u_α and $\bar{a}_{1,\alpha}^{(k)} = a_{1,\alpha}$ for all k by Lemma 6.5, $\bar{b}_{1,\alpha}^{(k)}$ must be strictly decreasing in view of Lemma 6.4. Similarly $\underline{b}_{1,\alpha}^{(k)}$ is strictly increasing in k. It remains to show that these two sequences have the same limit $b_{1,\alpha}$. This follows directly from the fact that $\bar{u}_\alpha^{(k)} \to u_\alpha$ and $\underline{u}_\alpha^{(k)} \to u_\alpha$ as $k \to +\infty$ once we show that the error terms in the expressions (6.15)–(6.17) have a uniform bound in k. To obtain such a uniform bound we proceed as follows.

As in Section 4, it is convenient to do the estimates in the variable $t = \ln r$ and for the function $W_\alpha^{(k)}(t) = r^{-\delta}\bar{u}_\alpha^{(k)}(r) - L$. Since $W_\alpha^{(k)}$ is uniformly bounded above and below, respectively by $W_\alpha^{(1)}$ and $W_\alpha(t) = r^{-\delta}u_\alpha(r) - L$ for all k and $\bar{a}_{1,\alpha}^{(k)} = a_{1,\alpha}$, we have, from (6.15) and (6.16) that

$$\begin{aligned} a_{1,\alpha}e^{-\lambda_1 t} - C_1 e^{-\lambda_2 t} &\leqslant W_\alpha(t) \leqslant W_\alpha^{(k)}(t) \leqslant W_\alpha^{(1)}(t) \\ &\leqslant a_{1,\alpha}e^{-\lambda_1 t} + C_1 e^{-\lambda_2 t} \end{aligned}$$

in $t \geqslant T_0$ where $C_1 > 0$ and $T_0 > 0$ are independent of k. That is, for $\omega > 0$ sufficiently small

$$\begin{aligned} 0 &< a_{1,\alpha}e^{-\lambda_1 t} - C_1 e^{-\lambda_2 t} \\ &\leqslant W_\alpha^{(k)}(t) \leqslant a_{1,\alpha}e^{-\lambda_1 t} + C_1 e^{-\lambda_2 t} < \omega \end{aligned}$$

in $t \geqslant T_0$ if T_0 is large enough. (Recall that $a_{1,\alpha} > 0$ by Lemma 6.5.) By the definition of g in (4.24), we know that $g(\tau)$ is increasing in $(0, \omega)$ for $\omega > 0$ sufficiently small. Thus, for $t \geqslant T_0$, by (4.24),

$$g\big(W_\alpha^{(k)}(t)\big) \leqslant g\big(a_{1,\alpha}e^{-\lambda_1 t} + C_1 e^{-\lambda_2 t}\big) \leqslant C_2 e^{-2\lambda_1 t}, \tag{6.20}$$

where $C_2 > 0$ is also independent of k,

$$g\big(W_\alpha^{(k)}(t)\big) \geqslant 0 \geqslant -C_3 e^{-2\lambda_1 t} \tag{6.21}$$

for $t \geqslant T_0$ and the constant $C_3 > 0$ is independent of k. Substituting (6.20) and (6.21) into (4.23) we obtain, after some computations as in Section 4, that

$$\big|W_\alpha^{(k)}(t) - \bar{a}_{1,\alpha}^{(k)} e^{-\lambda_1 t} - \bar{b}_{1,\alpha}^{(k)} e^{-\lambda_2 t}\big| \leqslant C_4 e^{-2\lambda_1 t} \tag{6.22}$$

for $t \geqslant T_0$, where the constant $C_4 > 0$ is independent of k.

Now suppose that $\lim_{k\to+\infty}\bar{b}_{1,\alpha}^{(k)} \neq b_{1,\alpha}$. Then there exists $\epsilon > 0$ such that $\bar{b}_{1,\alpha}^{(k)} > b_{1,\alpha} + \epsilon$ for k large. From (6.22) it follows that for $t \geqslant T_0$,

$$\begin{aligned}\bigl|\bigl(\bar{b}_{1,\alpha}^{(k)} - b_{1,\alpha}\bigr)e^{-\lambda_2 t}\bigr| &\leqslant \bigl|\bigl(\bar{a}_{1,\alpha}^{(k)}e^{-\lambda_1 t} + \bar{b}_{1,\alpha}^{(k)}e^{-\lambda_2 t}\bigr) - W_\alpha^{(k)}(t)\bigr| \\ &\quad + \bigl|W_\alpha^{(k)}(t) - W_\alpha(t)\bigr| + \bigl|W_\alpha(t) - \bigl(a_{1,\alpha}e^{-\lambda_1 t} + b_{1,\alpha}e^{-\lambda_2 t}\bigr)\bigr| \\ &\leqslant C_5 e^{-2\lambda_1 t} + \bigl|W_\alpha^{(k)}(t) - W_\alpha(t)\bigr|, \end{aligned} \tag{6.23}$$

where the constant $C_5 > 0$ is independent of k. Since $\lambda_2 < 2\lambda_1$ (by our assumption $\nu_2(n) < \nu < \nu_1(n)$), there exists a number $T_1 > T_0$ such that $e^{(2\lambda_1-\lambda_2)T_1} > C_5\epsilon^{-1}$. Letting $k \to +\infty$ in (6.23) (note that (6.7) holds) we obtain

$$\epsilon e^{-\lambda_2 T_1} \leqslant C_5 e^{-2\lambda_1 T_1}$$

which contradicts the choice of T_1. Therefore, $\bar{b}_{1,\alpha}^{(k)} \to b_{1,\alpha}$ as $k \to +\infty$. Similarly, $\underline{b}_{1,\alpha}^{(k)} \to b_{1,\alpha}$ as $k \to +\infty$, and our proof is complete. □

Now the stability of u_α in the norm $\|\cdot\|_{\lambda_2-\delta}$ is easily established by using Lemma 6.6. For given $\epsilon > 0$, Lemma 6.6 and estimate (6.22) guarantee that there exists k' such that if $\underline{u}_\alpha^{(k')} \leqslant v \leqslant \bar{u}_\alpha^{(k')}$ then

$$\|v - u_\alpha\|_{\lambda_2-\delta} < \epsilon.$$

On the other hand, for this k', since $\bar{u}_\alpha^{(k')} > u_\alpha > \underline{u}_\alpha^{(k')}$ in $\mathbf{R}^n$ and $\bar{b}_{1,\alpha}^{(k')} > b_{1,\alpha} > \underline{b}_{1,\alpha}^{(k')}$, there exists $\theta > 0$ such that if $\|\phi - u_\alpha\|_{\lambda_2-\delta} < \theta$ then $\bar{u}_\alpha^{(k')} > \phi > \underline{u}_\alpha^{(k')}$ in $\mathbf{R}^n$. Then Lemma 2.2 implies that $\bar{u}_\alpha^{(k')} > u(\cdot,t;\phi) > \underline{u}_\alpha^{(k')}$ in $\mathbf{R}^n$ for all $t > 0$ and therefore $\|u(\cdot,t;\phi) - u_\alpha\|_{\lambda_2-\delta} < \epsilon$. Thus u_α is stable with respect to the norm $\|\cdot\|_{\lambda_2-\delta}$.

To establish the weak asymptotic stability of u_α with respect to the norm $\|\cdot\|_{\lambda_2-\delta}$, it remains to show that the existence $\theta > 0$ such that for $\|\phi - u_\alpha\|_{\lambda_2-\delta} < \theta$ we always have $\|u(\cdot,t;\phi) - u_\alpha\|_{\lambda'} \to 0$ as $t \to +\infty$ for every $\lambda' < \lambda_2 - \delta$. This follows from Theorem 6.3 and Lemma 6.5 almost immediately. For we may choose $\theta > 0$ so small that if $\|\phi - u_\alpha\|_{\lambda_2-\delta} < \theta$ then $\underline{u}_\alpha^{(1)} \leqslant \phi \leqslant \bar{u}_\alpha^{(1)}$ in $\mathbf{R}^n$. Then Lemma 2.2 implies that

$$\underline{u}_\alpha^{(1)} < u\bigl(\cdot,t;\underline{u}_\alpha^{(1)}\bigr) < u(\cdot,t;\phi) < u\bigl(\cdot,t;\bar{u}_\alpha^{(1)}\bigr) < \bar{u}_\alpha^{(1)} \quad \text{in } \mathbf{R}^n. \tag{6.24}$$

Since u_α is the only steady-state satisfying (6.11) and both $u(\cdot,t;\bar{u}_\alpha^{(1)})$ and $u(\cdot,t;\underline{u}_\alpha^{(1)})$ are monotone in t, we must have

$$\lim_{t\to+\infty} u\bigl(x,t,;\bar{u}_\alpha^{(1)}\bigr) = u_\alpha(x) = \lim_{t\to+\infty} u\bigl(x,t;\underline{u}_\alpha^{(1)}\bigr).$$

Therefore, $u(\cdot,t;\phi) \to u_\alpha$ as $t \to +\infty$. Then, for every $\lambda' < \lambda_2 - \delta$ and every $R > 0$ it follows from (6.24) and the expansions (6.15)–(6.17) that

$$\begin{aligned}
&\left|\left(1+|x|\right)^{\lambda'}\left(u(x,t;\phi)-u_\alpha(x)\right)\right| \\
&\quad\leqslant \begin{cases} C(1+|x|)^{\lambda'}|x|^{\delta-\lambda_2} & \text{if } |x|\geqslant R, \\ (1+R)^{\lambda'}\|u(\cdot,t;\phi)-u_\alpha\|_{L^\infty(B_R)} & \text{if } |x|\leqslant R, \end{cases} \\
&\quad\leqslant \begin{cases} CR^{\lambda'-(\lambda_2-\delta)} & \text{if } |x|\geqslant R, \\ (1+R)^{\lambda'}\|u(\cdot,t;\phi)-u_\alpha\|_{L^\infty(B_R)} & \text{if } |x|\leqslant R. \end{cases}
\end{aligned}$$

Letting $t\to+\infty$ we obtain

$$\lim_{t\to+\infty}\sup\left\|u(\cdot,t;\phi)-u_\alpha\right\|_{\lambda'}\leqslant CR^{\lambda'-(\lambda_2-\delta)}.$$

Since R is arbitrary, we conclude that $\|u(\cdot,t;\phi)-u_\alpha\|_{\lambda'}\to 0$ as $t\to+\infty$. Therefore, u_α is weakly asymptotically stable with respect to the norm $\|\cdot\|_{\lambda_2-\delta}$ and the proof of Theorem 6.1 in the case $(\nu_c=)\nu_1(n)>\nu>\nu_2(n)$ is complete.

The rest of part (ii) of Theorem 6.1 can be handled in an analogous way. One simply notices that in Theorem 4.5 the coefficients $a_1,a_2,\dots,a_N$ are uniquely determined by a_1 and thus create no extra difficulties in extending Lemmas 6.4–6.6 to the more general case $\nu<\nu_1(n)=\nu_c$.

In proving part (i) of Theorem 6.1 by the above arguments, we first notice that now the two independent terms are $a_1r^{\delta-\lambda_1}\ln r$ and $b_1r^{\delta-\lambda_2}$ (which accounts for the slightly different norms used in (i)). As a result of this difference, (6.19) now takes a new form

$$a_{1,\alpha}=\alpha^{(\nu+1)\lambda_1/2}a_{1,1}\quad\text{and}\quad b_{1,\alpha}=\alpha^{(\nu+1)\lambda_1/2}\left(b_{1,1}-\frac{\nu+1}{2}a_{1,1}\ln\alpha\right).$$

Since the explicit form of $b_{1,\alpha}$ in (6.19) was never used in our proof of part (ii), this also causes no additional problem, and part (i) of Theorem 6.1 can now be established by the same arguments we used to handle the case $\nu_2(n)<\nu<\nu_1(n)$ earlier in this case. This completes the proof of Theorem 6.1. □

7. Expansion rate

In this section, we obtain the expansion rate of global solutions of (5.1) in some special cases.

Theorem 7.1. *Suppose $\nu>0$ and $\psi\in C_{LB}(\mathbf{R}^N)$ is a c.w. sub-solution of* (4.1). *If the initial value $\phi\geqslant\gamma\psi$ for some $\gamma>1$, then* (5.1) *has a unique global classical solution u satisfying $\gamma\psi\leqslant u\leqslant e^{t\Delta}\phi$ and for $t>0$*

$$\min_{\mathbf{R}^n}u(\cdot,t)\geqslant(\nu+1)^{1/(\nu+1)}\left(\gamma^{\nu+1}-1\right)^{1/(\nu+1)}t^{1/(\nu+1)}.$$

Proof. The global existence follows from Lemma 2.2 if we can show $\gamma\psi\leqslant e^{t\Delta}\phi$ in $\mathbf{R}^n\times[0,+\infty)$. But this can be obtained from Lemma 2.3. (It is clear that $e^{t\Delta}\phi$ is a super-solution of (5.1).) The uniqueness can also be obtained from Lemma 2.3. To prove the large time behavior of u, it suffices to take $\phi=\gamma\psi$. First, we assume ψ is C^∞ smooth, then u is C^∞ smooth to the boundary $t=0$. Consider $v=u_t-\theta u^{-\nu}$ where constant $\theta>0$ is to be determined later. By a straightforward computation we have

$$v_t - \Delta v \geqslant \nu u^{-(\nu+1)} v \quad \text{on } \mathbf{R}^n \times [0, +\infty).$$

Observe that

$$\begin{aligned} v|_{t=0} &= \left(u_t - \theta u^{-\nu}\right)\big|_{t=0} = \left(\Delta u - (\theta+1)u^{-\nu}\right)\big|_{t=0} \\ &= \gamma \Delta\psi - (\theta+1)\gamma^{-\nu}\psi^{-\nu} \\ &\geqslant \gamma\psi^{-\nu}\left[1 - (\theta+1)\gamma^{-(\nu+1)}\right] \\ &= 0 \quad \text{if } \theta = \gamma^{\nu+1} - 1. \end{aligned}$$

From Lemma 3.4, $u_t \geqslant 0$. So, $v \geqslant -\theta u^{-\nu} \geqslant -\theta\psi^{-\nu}$. In particular, $\nu u^{-(\nu+1)}(\cdot, t)$ has a positive lower bound for any $t > 0$. Then by the Phragmèn–Lindelöf comparison principle (see Lemma 2.3), $v \geqslant 0$, i.e., $u_t \geqslant \theta u^{-\nu}$ with $\theta = \gamma^{\nu+1} - 1$. This in turn implies that

$$u(x, t) \geqslant (\nu+1)^{1/(\nu+1)}\left(\gamma^{\nu+1} - 1\right)^{1/(\nu+1)} t^{1/(\nu+1)} \quad \text{for } t > 0,$$

and this completes the argument for regular ψ.

For the general case, consider the global classical solution u_ψ of (5.1) with $\phi = \psi$ (u_ψ is assured by Lemma 2.2 again). By uniqueness, this u_ψ is the same one as in Theorem 3.3. Hence by the proof of Theorem 3.3, $u_\psi(\cdot, t) \to \psi(\cdot)$ pointwise as $t \to 0^+$. Also, by Lemma 3.4, $\partial u_\psi/\partial t \geqslant 0$ for $t > 0$ and hence $\psi_\epsilon(\cdot) = u_\psi(\cdot, \epsilon)$ is a smooth (by regularity theory) sub-solution of (4.1). Therefore, the conclusion for smooth ψ implies that

$$u_\epsilon(x, t) \geqslant (\nu+1)^{1/(\nu+1)}\left(\gamma^{\nu+1} - 1\right)^{1/(\nu+1)} t^{1/(\nu+1)},$$

where u_ϵ is the global classical solution of (5.1) and $\phi = \gamma\psi_\epsilon$. We claim that $u_\epsilon \to u$ pointwise on $\mathbf{R}^n \times [0, +\infty)$ (hence we are done). In fact, this follows from the continuity of solutions with respect to the initial value. This continuity can be proved by the integral equation and Gronwall inequality. This completes the proof of Theorem 7.1. □

Corollary 7.2. *Suppose that $\nu > 0$. If the initial value $\phi \geqslant \gamma u_s$ for some constant $\gamma > 1$, then* (5.1) *has a unique global classical solution u satisfying $u \geqslant \gamma u_s$ and*

$$\min_{\mathbf{R}^n} u(\cdot, t) \geqslant (\nu+1)^{1/(\nu+1)}\left(\gamma^{\nu+1} - 1\right)^{1/(\nu+1)} t^{1/(\nu+1)}.$$

Proof. Again, the uniqueness immediately follows from the Phragmèn–Lindelöf comparison principle (see Lemma 2.3). On the other hand, exactly as in the proof of (iii) of Theorem 5.6, we can find a c.w. sub-solution ψ of (4.1) such that $\phi \geqslant \gamma\psi \geqslant \gamma u_s$ when $\nu > \nu_c$, and $\phi \geqslant \gamma'\psi \geqslant \gamma' u_s$ when $0 < \nu \leqslant \nu_c$, where γ' and ψ can be chosen so that γ' can be arbitrarily close to γ and $\gamma > \gamma' > 1$. By Theorem 7.1, in any case, (5.1) has a unique global classical solution u so that $u \geqslant \gamma'\psi$ ($\geqslant \gamma' u_s$) and

$$\min_{\mathbf{R}^n} u(\cdot, t) \geqslant (\nu+1)^{1/(\nu+1)}\left((\gamma')^{\nu+1} - 1\right)^{1/(\nu+1)} t^{1/(\nu+1)}.$$

Letting $\gamma' \to \gamma$, we are done. This completes the proof of this corollary. □

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